

$$iv) \text{ let } \tanh(1+i) = x+iy$$

$$x+iy = \frac{\sinh(1+i)}{\cosh(1+i)}$$

$$\sinh x = \frac{1}{i} \sin(ix)$$

$$\cosh x = \cos(ix)$$

$$= \frac{\frac{1}{i} \sin(i(1+i))}{\cos(i(1+i))}$$

$$= \frac{-i \sin(i-1)}{\cos(i-1)}$$

$$= \frac{-i \sin(i-1)}{\cos(i-1)} \cdot \frac{\cos(i+1)}{\cos(i+1)}$$

$$= -i \frac{2 \sin(i-1) \cdot \cos(i+1)}{2 \cos(i-1) \cdot \cos(i+1)}$$

$$\therefore 2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$x+iy = -i \frac{(\sin(i-1+i+1) + \sin(i-1-(i+1)))}{(\cos(i-1-(i+1)) + \cos(i-1+i+1))}$$

$$= -i \frac{[\sin 2i + \sin(-2)]}{[\cos(-2) + \cos(2i)]}$$

$$= -i \left[\frac{\sin 2i + \sin(-2)}{\cos(-2) + \cos(2i)} \right]$$

$$= \frac{-i [\sin a i - \sin a]}{\cos a + \cos a i}$$

$$\begin{aligned} \therefore \sin(-0) &= -\sin 0 \\ \cos(-0) &= \cos 0 \end{aligned}$$

$$= \frac{-i (i \sinh a - \sin a)}{\cos a + \cosh a}$$

$$\sinh x = \frac{1}{i} \sin(ix)$$

$$\cosh x = \cos(ix)$$

$$= \frac{\sinh a + i \sin a}{\cos a + \cosh a}$$

$$x + iy = \frac{\sinh a}{\cos a + \cosh a} + i \frac{\sin a}{\cos a + \cosh a}$$

$$\therefore \text{Real part} = \frac{\sinh a}{\cos a + \cosh a} \text{ and}$$

$$\text{Imaginary part} = \frac{\sin a}{\cos a + \cosh a}$$

13) If $\cos \alpha + i \sin \alpha = \cos(\theta + i\phi)$ prove that $\sin^2 \theta = \pm \sin \alpha$

Soln.

$$\cos \alpha + i \sin \alpha = \cos(\theta + i\phi)$$

$$= \cos \theta \cos(i\phi) - \sin \theta \sin(i\phi)$$

$$= \cos \theta \cosh \phi - \sin \theta i \sinh \phi$$

$$\begin{aligned} \therefore \sinh x &= \frac{1}{i} \sin(ix) \\ \cosh x &= \cos(ix) \end{aligned}$$

$$\cos \alpha + i \sin \alpha = \cos \theta \cosh \phi - i \sin \theta \sinh \phi$$

Equating real and imaginary parts, we get

$$\cos \alpha = \cos \theta \cosh \phi \quad \text{--- ①}$$

$$\sin \alpha = -\sin \theta \sinh \phi \quad \text{--- ②}$$

From ① and ②

$$\cosh \phi = \frac{\cos \alpha}{\cos \theta} \quad \text{and} \quad \sinh \phi = \frac{-\sin \alpha}{\sin \theta}$$

$$\cosh^2 \phi = \frac{\cos^2 \alpha}{\cos^2 \theta}, \quad \sinh^2 \phi = \frac{\sin^2 \alpha}{\sin^2 \theta}$$

$$\cosh^2 \phi - \sinh^2 \phi = \frac{\cos^2 \alpha}{\cos^2 \theta} - \frac{\sin^2 \alpha}{\sin^2 \theta}$$

$\therefore \cosh^2 \phi - \sinh^2 \phi$

$$= \frac{\cos^2 \alpha \sin^2 \theta - \sin^2 \alpha \cos^2 \theta}{\cos^2 \theta \sin^2 \theta}$$

$$\cos^2 \theta \sin^2 \theta = \cos^2 \alpha \sin^2 \theta - \sin^2 \alpha \cos^2 \theta$$

$$(1 - \sin^2 \theta) \sin^2 \theta = (1 - \sin^2 \alpha) \sin^2 \theta - \sin^2 \alpha (1 - \sin^2 \theta)$$

$$\sin^2 \theta - \sin^4 \theta = \sin^2 \theta - \sin^2 \alpha \sin^2 \theta - \sin^2 \alpha + \sin^2 \alpha \sin^2 \theta$$

$$\cancel{\sin^2 \theta} - \sin^4 \theta = \cancel{\sin^2 \theta} - \sin^2 \alpha$$

$$+ \sin^4 \theta = -\sin^2 \alpha$$

$$(\sin^2 \theta)^2 = \sin^2 \alpha$$

$$\boxed{\sin^2 \theta = \pm \sin \alpha}$$

14) If $\tan(\theta + i\phi) = \cos \alpha + i \sin \alpha$ prove that

$$i) \quad \theta = \frac{1}{2} n\pi + \frac{1}{4} \pi$$

$$ii) \quad \phi = \frac{1}{2} \log \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right)$$

Soln.

$$\text{Given } \tan(\theta + i\phi) = \cos \alpha + i \sin \alpha \quad \text{--- ①}$$

$$\therefore \tan(\theta - i\phi) = \cos\alpha - i\sin\alpha \quad \text{--- (2)}$$

(i) Now, $2\theta = (\theta + i\phi) + (\theta - i\phi)$

$$\tan 2\theta = \tan [(\theta + i\phi) + (\theta - i\phi)]$$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi) \cdot \tan(\theta - i\phi)}$$

$$= \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)}$$

$$= \frac{2\cos\alpha}{1 - 1}$$

$$\tan 2\theta = \infty$$

$$\tan 2\theta = \tan \frac{\pi}{2}$$

$$\tan 2\theta = \tan \left(n\pi + \frac{\pi}{2}\right)$$

$$2\theta = n\pi + \frac{\pi}{2}$$

$$\theta = \frac{1}{2}n\pi + \frac{\pi}{4}$$

$$\tan \frac{\pi}{2} = \infty$$

$$\tan \left(n\pi + \frac{\pi}{2}\right) = \tan \frac{\pi}{2}$$

$$(ii) \tan(2i\phi) = \tan[(\theta + i\phi) - (\theta - i\phi)]$$

$$i \tanh 2\phi = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi) \cdot \tan(\theta - i\phi)}$$

$$= \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1 + (\cos\alpha + i\sin\alpha) \cdot (\cos\alpha - i\sin\alpha)}$$

$$= \frac{\cancel{\cos\alpha} + i\sin\alpha - \cancel{\cos\alpha} + i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)}$$

$$(\because \cos^2\alpha + \sin^2\alpha = 1)$$

$$= \frac{2i\sin\alpha}{1+1}$$

$$i \tanh 2\phi = \frac{2i\sin\alpha}{2}$$

$$\tanh 2\phi = \sin\alpha$$

$$2\phi = \tanh^{-1}(\sin\alpha)$$

$$= \frac{1}{2} \log \left(\frac{1 + \sin\alpha}{1 - \sin\alpha} \right)$$

$$\sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

$$= \frac{1}{2} \log \left[\frac{1 + \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}}{1 - \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}} \right]$$

$$= \frac{1}{2} \log \left[\frac{\frac{1 + \tan^2 \frac{\alpha}{2} + 2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}}{\frac{1 + \tan^2 \frac{\alpha}{2} - 2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}} \right]$$



$$= \frac{1}{2} \log \frac{(1 + \tan \frac{\alpha}{2})}{(1 - \tan \frac{\alpha}{2})^2}$$

$$= \frac{1}{2} \log \left[\frac{1 + \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2}} \right]^2$$

$$2\phi = \log \left(\left[\frac{1 + \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2}} \right]^2 \right)^{\frac{1}{2}}$$

$$2\phi = \log \frac{1 + \tan \frac{\alpha}{2}}{1 - \tan \frac{\alpha}{2}} \quad \left(\because \tan \frac{\pi}{4} = 1 \right)$$

$$\phi = \frac{1}{2} \log \left(\frac{\tan \frac{\pi}{4} + \tan \frac{\alpha}{2}}{1 - \tan \frac{\pi}{4} \cdot \tan \frac{\alpha}{2}} \right)$$

$$\phi = \frac{1}{2} \log \left(\tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right)$$

Logarithm of Complex Numbers

Defn: Let $z = r(\cos \theta + i \sin \theta)$ be a non-zero Complex number. we define $\log z = \log r + i\theta$

Note: 1

Since the amplitude θ has infinite number of values, which differ by $2n\pi$ for some $n \in \mathbb{Z}$; $\log z$ has infinite number of values which differ by $2in\pi$. If we take θ to be the principal value

of the amplitude of z , the corresponding value of $\log z$ is called its principal value.

Hence if we denote the general values of $\log z$ by $\text{Log } z$, we have

$$\log z = \log z + i(2n\pi), \text{ where } n \in \mathbb{Z}$$

$$\text{ie) } \text{Log } z = \log r + i\theta + i(2n\pi), \text{ where } n \in \mathbb{Z}$$

$$(1 = \text{Principal Value}) = \log r + i(\theta + 2n\pi)$$

Note 2: let $z = x + iy = r(\cos\theta + i\sin\theta)$

$$\text{Hence } r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \log z = \log(\sqrt{x^2 + y^2}) + i \tan^{-1}\left(\frac{y}{x}\right)$$

Note 3:

If $z = x$ is real and positive, $\log z = \log z + i2n\pi$

where $n \in \mathbb{Z}$. Hence one value of $\log z$ is a positive real number and all other values are non-real.

Thm:

$$\log z_1 z_2 = \log z_1 + \log z_2$$

Proof: let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\text{Hence } z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \left[\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2 \right]$$

$$= r_1 r_2 \left[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \right]$$

$$z_1 z_2 = r_1 r_2 \left[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2) \right]$$

$$(\because \cos (\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$$

$$\sin (\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)$$

$$\log z_1 z_2 = \log r_1 r_2 + i (\theta_1 + \theta_2 + 2n\pi), n \in \mathbb{Z}$$

$$= \log r_1 + \log r_2 + i (\theta_1 + 2n\pi) + i (\theta_2 + 2m\pi)$$

where $m, n \in \mathbb{Z}$

$$= [\log r_1 + i (\theta_1 + 2n\pi)] + [\log r_2 + i (\theta_2 + 2m\pi)]$$

$$\log z_1 z_2 = \log z_1 + \log z_2$$

Defn:

If z and w are any two complex numbers then we define $z^w = e^{w \log z}$

Note:

Since $\log z$ has infinite number of values for z , z^w also has infinite number of values.

Prob 1: Prove that $\text{Log } i = i(4n+1)\left(\frac{\pi}{2}\right)$

Soln.

$$i = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)$$

w.k.t $z = r(\cos\theta + i\sin\theta)$

$$\log z = \log r + i(\theta + 2n\pi)$$

$$\log i = \log 1 + i\left[\frac{\pi}{2} + 2n\pi\right]$$

$$= 0 + i\left[\left(\frac{\pi}{2}\right) + 2n\pi\right]$$

$$= i\left[\frac{\pi + 4n\pi}{2}\right]$$

$$\log i = i(4n+1)\left(\frac{\pi}{2}\right)$$

Prob 2: Find (i) $\text{Log}(1+i)$ (ii) $\text{Log}(-e)$

Soln

w.k.t $z = x+iy = r(\cos\theta + i\sin\theta)$

$$\log z = \log \sqrt{x^2+y^2} + i \tan^{-1}\left(\frac{y}{x}\right), \text{ where } r = \sqrt{x^2+y^2},$$

Let $z = 1+i$, $r = \sqrt{1^2+1^2} = \sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1} 1$$

$$= \tan^{-1}\left(\tan \frac{\pi}{4}\right)$$

$$\boxed{\theta = \frac{\pi}{4}}$$

$$z = r (\cos \theta + i \sin \theta), \log z = \log r + i (\theta + 2n\pi)$$

$$1+i = \sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]$$

$$\begin{aligned} \log(1+i) &= \log \sqrt{2} + i (\theta + 2n\pi) \\ &= \log \sqrt{2} + i \left(\frac{\pi}{4} + 2n\pi \right) \end{aligned}$$

$$= \log(2)^{\frac{1}{2}} + i \left(\frac{\pi + 8n\pi}{4} \right)$$

$$\log(1+i) = \frac{1}{2} \log 2 + i (1+8n) \frac{\pi}{4}$$

$$(ii) -e = -e (\cos \pi + i \sin \pi)$$

$$\log(-e) = \log e + i (\pi + 2n\pi)$$

$$\log(-e) = 1 + i (2n+1)\pi$$

3) Prove that

$$\log \left(\frac{1}{1-e^{i\theta}} \right) = \log \left(\frac{\cos(\theta/2)}{2} \right) + i \left(2n\pi + \frac{\pi}{2} - \frac{\theta}{2} \right)$$

Soln

$$\begin{aligned} \frac{1}{1-e^{i\theta}} &= (1-e^{i\theta})^{-1} \\ &= [1 - (\cos \theta + i \sin \theta)]^{-1} \end{aligned}$$

$$\because e^{i\theta} = \cos \theta + i \sin \theta$$

$$= [(1 - \cos \theta) + i \sin \theta]^{-1}$$

$$= [\sin^2 \frac{\theta}{2} - i \sin \frac{\theta}{2} \cos \frac{\theta}{2}]^{-1}$$

$$= [\sin \frac{\theta}{2} (\sin \frac{\theta}{2} - i \cos \frac{\theta}{2})]^{-1} \quad \left(\begin{array}{l} \cos 2\theta = 1 - 2\sin^2 \theta \\ \cos \theta = 1 - 2\sin^2 \frac{\theta}{2} \\ 2\sin^2 \frac{\theta}{2} = 1 - \cos \theta \end{array} \right)$$

$$= (2 \sin \frac{\theta}{2})^{-1} (\sin \frac{\theta}{2} - i \cos \frac{\theta}{2})^{-1} \quad \sin 2\theta = 2 \sin \theta \cos \theta$$

$$= \frac{1}{2 \sin \frac{\theta}{2}} [\cos (\frac{\pi}{2} - \frac{\theta}{2}) - i \sin (\frac{\pi}{2} - \frac{\theta}{2})]^{-1}$$

$$\frac{1}{1 - e^{i\theta}} = \frac{1}{2} \operatorname{cosec} \frac{\theta}{2} [\cos (\frac{\pi}{2} - \frac{\theta}{2}) + i \sin (\frac{\pi}{2} - \frac{\theta}{2})]$$

$$\log \left(\frac{1}{1 - e^{i\theta}} \right) = \log \left(\frac{\operatorname{cosec} (\frac{\theta}{2})}{2} \right) + i \left(\frac{\pi}{2} - \frac{\theta}{2} \right) + i (2n\pi)$$

$$\left(\because \log z = \log r + i(\theta + 2n\pi) \right)$$

$$= \log \left(\frac{\operatorname{cosec} (\frac{\theta}{2})}{2} \right) + i \left(2n\pi + \frac{\pi}{2} - \frac{\theta}{2} \right)$$

Pr. 4 Prove that $i^i = e^{-(4n+1)(\frac{\pi}{2})}$

Soln. $i^i = e^{i \log i}$ (by definition)

$$\text{w.k.t } \log i = i(4n+1)(\frac{\pi}{2})$$

$$i^i = e^{i(i(4n+1)(\frac{\pi}{2}))}$$

$$= e^{-(4n+1)\frac{\pi}{2}}$$

5) If $i^{a+ib} = a+ib$ prove that $a^2+b^2=e$ - (4n+1)\pi b.

Soln.

Given $i^{a+ib} = a+ib$ ——— ①

$$i^{a+ib} = e^{(a+ib) \log i}$$

$$= e^{(a+ib) i (4n+1) (\frac{\pi}{2})}$$

$$(\because \log i = i (4n+1) (\frac{\pi}{2}))$$

$$i^{a+ib} = e^{(ai-b)(4n+1) \frac{\pi}{2}}$$

$$= e^{ai(4n+1) \frac{\pi}{2} - b(4n+1) \frac{\pi}{2}}$$

$$= e^{ai(4n+1) \frac{\pi}{2}} \cdot e^{-b(4n+1) \frac{\pi}{2}}$$

$$= e^{-b(4n+1) \frac{\pi}{2}} \cdot e^{i\theta}$$

$$= e^{-b(4n+1) \frac{\pi}{2}} (\cos \theta + i \sin \theta), \text{ where } \theta = a(4n+1) \frac{\pi}{2}$$

$$i^{a+ib} = e^{-b(4n+1) \frac{\pi}{2}} (\cos \theta + i \sin \theta)$$

$\therefore$ By eqn ①, $a+ib = e^{-b(4n+1) \frac{\pi}{2}} (\cos \theta + i \sin \theta)$

$$a+ib = e^{-b(4n+1) \frac{\pi}{2}} \cos \theta + i e^{-b(4n+1) \frac{\pi}{2}} \sin \theta$$

Equating real and imaginary parts.

$$a = e^{-b(4n+1) \frac{\pi}{2}} \cos \theta.$$

$$b = e^{-b(4n+1) \frac{\pi}{2}} \sin \theta.$$



$$a^2 = \left(e^{-b(4n+1)\frac{\pi}{2}} \cos \theta \right)^2 = e^{-b(4n+1) \cdot \frac{\pi}{2} \cdot 2} \cos^2 \theta$$

$$a^2 = e^{-b(4n+1) \cdot \pi} \cdot \cos^2 \theta$$

$$b^2 = \left(e^{-b(4n+1)\left(\frac{\pi}{2}\right)} \sin \theta \right)^2 = e^{-b(4n+1) \cdot \pi} \sin^2 \theta$$

$$a^2 + b^2 = e^{-b(4n+1) \cdot \pi} \cdot \cos^2 \theta + e^{-b(4n+1) \pi} \sin^2 \theta$$

$$= e^{-b(4n+1) \pi} (\cos^2 \theta + \sin^2 \theta)$$

$$a^2 + b^2 = e^{-b(4n+1) \pi}$$

$$a^2 + b^2 = e^{-(4n+1) \pi b}$$
