

3) If $\tan(a+ib) = x+iy$ prove that

$$\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$$

Soln.

Given $x+iy = \tan(a+ib)$

$$= \frac{\sin(a+ib)}{\cos(a+ib)}$$

$$(\because \tan x = \frac{\sin x}{\cos x})$$

$$= \frac{\sin(a+ib)}{\cos(a+ib)} \cdot \frac{\cos(a-ib)}{\cos(a-ib)}$$

$$x+iy = \frac{2(\sin(a+ib) \cdot \cos(a-ib))}{2(\cos(a+ib) \cdot \cos(a-ib))}$$

Formula
 $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$$2 \cos A \cos B = \cos(A-B) + \cos(A+B)$$

$$x+iy = \frac{\sin(a+ib+a-ib) + \sin(a+ib-a-ib)}{\cos(a+ib-a-ib) + \cos(a+ib+a-ib)}$$

$$= \frac{\sin 2a + \sin(2ib)}{\cos(2ib) + \cos(2a)} = \frac{\sin 2a + \sin i(2b)}{\cos i(2b) + \cos 2a}$$

$$= \frac{\sin 2a + i \sinh 2b}{\cosh 2b + \cos 2a}$$

$$\begin{aligned} (\because \sin(ix) &= i \sinh x \\ \cos ix &= \cosh x) \end{aligned}$$

$$x+iy = \frac{\sin 2a}{\cos 2a + \cosh 2b} + i \frac{\sinh 2b}{\cos 2a + \cosh 2b}$$

Equating real and Imaginary parts

$$x = \frac{\sin 2a}{\cos 2a + \cosh 2b} \quad \text{and} \quad y = \frac{\sinh 2b}{\cos 2a + \cosh 2b}$$

$$\frac{x}{y} = \frac{\frac{\sin 2a}{\cos 2a + \cosh 2b}}{\frac{\sinh 2b}{\cos 2a + \cosh 2b}}$$

$$\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$$

4) If $x+iy = \tan(A+iB)$ prove that

$$x^2 + y^2 + 2x \cot 2A = 1$$

Soln

$$\text{Given } x+iy = \tan(A+iB) \quad \text{--- (1)}$$

$$x-iy = \tan(A-iB) \quad \text{--- (2)}$$

$$\text{Now, } \cot 2A = \frac{1}{\tan 2A}$$

$$\begin{aligned} &= \frac{1}{\tan[(A+iB)+(A-iB)]} \quad \tan(A+B) \\ &= \frac{1}{\tan(A+iB) + \tan(A-iB)} \quad = \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{1}{1 - \tan(A+iB) \tan(A-iB)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1 - \tan(A+iB) \tan(A-iB)}{\tan(A+iB) + \tan(A-iB)} \\
 &= \frac{1 - (x+iy)(x-iy)}{(x+iy) + (x-iy)} \quad (\because \text{eqn (1) \& (2)}) \\
 &= \frac{1 - (x^2 - (iy)^2)}{x+iy + x-iy} \\
 \cos 2A &= \frac{1 - (x^2 + y^2)}{2x}
 \end{aligned}$$

$$\begin{aligned}
 2x \cot 2A &= 1 - x^2 - y^2 \\
 x^2 + y^2 + 2x \cot 2A &= 1.
 \end{aligned}$$

5) If $x+iy = \sin(A+iB)$ prove that

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1.$$

Soln. Given $x+iy = \sin(A+iB)$

$$x+iy = \sin A \cos(iB) + \cos A \sin(iB)$$

$$= \sin A \cosh B + \cos A (i \sinh B)$$

$$(\because \sin(ix) = i \sinh x)$$

$$\cos(ix) = \cosh x$$

$$x+iy = \sin A \cosh B + i (\cos A \sinh B)$$

Equating real and imaginary parts, we get

$$x = \sin A \cosh B, \quad y = \cos A \sinh B$$

$$\frac{x}{\sin A} = \cosh B, \quad \frac{y}{\cos A} = \sinh B$$

$$\frac{x^2}{\sin^2 A} = \cosh^2 B, \quad \frac{y^2}{\cos^2 A} = \sinh^2 B$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

6) If $\cos(x+iy) = \cos \theta + i \sin \theta$ prove that

$$\cos 2x + \cosh 2y = 2.$$

Soln

$$\cos \theta + i \sin \theta = \cos(x+iy)$$

$$= \cos x \cos(iy) - \sin x \sin(iy)$$

$$= \cos x \cosh y - \sin x (i \sinh y)$$

$$\begin{cases} \cos ix = \cosh x \\ \sin ix = i \sinh x \end{cases}$$

$$\cos \theta + i \sin \theta = \cos x \cosh y - i \sin x \sinh y$$

Equating real and imaginary parts, we get

$$\cos \theta = \cos x \cosh y, \quad \text{and} \quad \sin \theta = -\sin x \sinh y$$

Squaring and adding we get

$$\cos^2 \theta = \cos^2 x \cosh^2 y, \quad \sin^2 \theta = \sin^2 x \sinh^2 y$$

$$\cos^2 \theta + \sin^2 \theta = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$

$$1 = \cos^2 x \cosh^2 y + (1 - \cos^2 x) \sinh^2 y$$

$$(\sin^2 x = 1 - \cos^2 x)$$

$$= \cos^2 x \cosh^2 y + \sinh^2 y - \cos^2 x \sinh^2 y$$

$$= \cos^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y$$

$$= \cos^2 x (1) + \sinh^2 y$$

$$1 = \cos^2 x + \sinh^2 y$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$1 = \frac{1 + \cos 2x}{2} + \frac{\cosh 2y - 1}{2}$$

$$2 = 1 + \cos 2x + \cosh 2y - 1$$

$$\cos 2x + \cosh 2y = 2.$$

7) If $\sin(\theta + i\phi) = \tan \alpha + i \sec \alpha$ prove that

$$\cos 2\theta \cosh 2\phi = 3.$$

Soln Given $\sin(\theta + i\phi) = \tan \alpha + i \sec \alpha$

$$\text{ie) } \sin \theta \cos(i\phi) + \cos \theta \sin(i\phi) = \tan \alpha + i \sec \alpha$$

$$\text{ie) } \sin \theta \cosh \phi + \cos \theta (i \sinh \phi) = \tan \alpha + i \sec \alpha$$

$$\sin \theta \cosh \phi + i \cos \theta \sinh \phi = \tan \alpha + i \sec \alpha$$

Equating real and imaginary parts, we get

$$\sin \theta \cosh \phi = \tan \alpha \quad \& \quad \cos \theta \sinh \phi = \sec \alpha$$

$$\sin^2 \theta \cosh^2 \phi = \tan^2 \alpha \quad \& \quad \cos^2 \theta \sinh^2 \phi = \sec^2 \alpha \quad \text{--- (1)}$$

Using the identity $1 + \tan^2 \alpha = \sec^2 \alpha$ we get

$$1 + \sin^2 \theta \cosh^2 \phi = 1 + \tan^2 \alpha \text{ \& \; } \cos^2 \theta \sinh^2 \phi = \sec^2 \alpha$$

$$1 + \sin^2 \theta \cosh^2 \phi = \sec^2 \alpha,$$

$$1 + \sin^2 \theta \cosh^2 \phi = \cos^2 \theta \sinh^2 \phi$$

$$1 + (1 - \cos^2 \theta) \cosh^2 \phi = \cos^2 \theta \sinh^2 \phi$$

$$1 + \cosh^2 \phi - \cos^2 \theta \cosh^2 \phi = \cos^2 \theta \sinh^2 \phi$$

$$1 + \cosh^2 \phi - \cos^2 \theta \cosh^2 \phi - \cos^2 \theta \sinh^2 \phi = 0$$

$$1 + \cosh^2 \phi - \cos^2 \theta (\cosh^2 \phi + \sinh^2 \phi) = 0$$

$$1 + \frac{1 + \cosh 2\phi}{2} - \cos^2 \theta (\cosh 2\phi) = 0$$

$$1 + \frac{1}{2} (1 + \cosh 2\phi) - \frac{1}{2} (1 + \cos 2\theta) \cosh 2\phi = 0$$

$$\frac{2 + 1 + \cosh 2\phi - (1 + \cos 2\theta) \cosh 2\phi}{2} = 0$$

$$3 + \cosh 2\phi - \cosh 2\phi - \cos 2\theta \cosh 2\phi = 0$$

$$\cos 2\theta \cosh 2\phi = 3$$

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \sin^2 \theta &= 1 - \cos^2 \theta \end{aligned}$$

$$\cosh 2\phi = \cosh^2 \phi + \sinh^2 \phi$$

$$\cosh 2x = 2 \cosh^2 x - 1$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

$$\cos^2 \theta = \frac{\cos 2\theta + 1}{2}$$

8) If $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$ prove that

$$y = \frac{1}{2} \log \left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)} \right]$$

Soln Given

$$\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos x \cos(iy) - \sin x \sin(iy) = r(\cos\alpha + i\sin\alpha)$$

$$\cos x \cosh y - \sin x (i \sinh y) = r \cos\alpha + i r \sin\alpha$$

$$\cos x \cosh y - i \sin x \sinh y = r \cos\alpha + i r \sin\alpha$$

Equating real and imaginary parts, we get

$$\cos x \cosh y = r \cos\alpha \rightarrow (1) \text{ and } -\sin x \sinh y = r \sin\alpha \rightarrow (2)$$

Divide eqn (2) by eqn (1)

$$\frac{r \sin\alpha}{r \cos\alpha} = \frac{-\sin x \sinh y}{\cos x \cosh y}$$

$$\tan\alpha = - \left(\frac{\sin x}{\cos x} \right) \left(\frac{\sinh y}{\cosh y} \right)$$

$$\tan\alpha = - \tanh x \tanh y..$$

$$\text{Hence } \tanh y = - \frac{\tanh\alpha}{\tanh x}$$

$$y = \tanh^{-1} \left(\frac{-\tanh\alpha}{\tanh x} \right)$$

$$\text{w.k.t } \tanh^{-1} x = \frac{1}{2} \log \left| \frac{1+x}{1-x} \right|$$

$$y = \frac{1}{2} \log \left[\frac{1 + \left(\frac{-\tan \alpha}{\tan x} \right)}{1 - \left(\frac{-\tan \alpha}{\tan x} \right)} \right]$$

$$= \frac{1}{2} \log \left[\frac{1 - \frac{\tan \alpha}{\tan x}}{1 + \frac{\tan \alpha}{\tan x}} \right]$$

$$= \frac{1}{2} \log \left[\frac{\frac{\tan x - \tan \alpha}{\tan x}}{\frac{\tan x + \tan \alpha}{\tan x}} \right]$$

$$= \frac{1}{2} \log \left[\frac{\tan x - \tan \alpha}{\tan x + \tan \alpha} \right]$$

$$= \frac{1}{2} \log \left[\frac{\frac{\sin x}{\cos x} - \frac{\sin \alpha}{\cos \alpha}}{\frac{\sin x}{\cos x} + \frac{\sin \alpha}{\cos \alpha}} \right]$$

$$= \frac{1}{2} \log \left[\frac{\frac{\sin x \cos \alpha - \sin \alpha \cos x}{\cos x \cos \alpha}}{\frac{\sin x \cos \alpha + \sin \alpha \cos x}{\cos x \cos \alpha}} \right]$$

$$= \frac{1}{2} \log \left[\frac{\sin x \cos \alpha - \sin \alpha \cos x}{\sin x \cos \alpha + \sin \alpha \cos x} \right]$$

$$y = \frac{1}{2} \log \frac{\sin(x-\alpha)}{\sin(x+\alpha)}$$

9) If $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{x}{2}\right)$ prove that

$$\cos x \cosh x = 1.$$

Soln.

$$\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$$

$$\cosh x = \frac{1 + \tanh^2\left(\frac{x}{2}\right)}{1 - \tanh^2\left(\frac{x}{2}\right)} \quad \text{--- (1)}$$

$$\text{Given } \tan\left(\frac{x}{2}\right) = \tanh\left(\frac{x}{2}\right).$$

$$(1) \Rightarrow \cosh x = \frac{1 + \tan^2 \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$$

$$\cosh x = \frac{1}{\cos x}$$

$$\cos x \cosh x = 1$$

10) Prove $u = \log_e \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ if and only if

$$\cosh u = \sec \theta.$$

Soln.

$$\text{Let } \cosh u = \sec \theta$$

$$u = \cosh^{-1}(\sec \theta)$$