

Summation of Trigonometric Series

$$S_n = \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n-1)\beta)$$

$$S_n = \frac{\sin\left(\alpha + \frac{(n-1)\beta}{2}\right) \cdot \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

$$S_n = \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (n-1)\beta)$$

$$S_n = \frac{\cos\left(\alpha + \frac{(n-1)\beta}{2}\right) \cdot \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

3) Find $S_n = \sin \alpha - \sin 2\alpha + \sin 3\alpha - \sin 4\alpha + \dots$

n terms

Sln

W.k.t $\sin(\pi + 2\alpha) = -\sin 2\alpha$

$\sin(2\pi + 3\alpha) = \sin 3\alpha$

$\sin(3\pi + 4\alpha) = -\sin 4\alpha$

	90	
sin	cos	All
cos	sin	
tan	cot	
cot	tan	
	270	

$$\begin{aligned} \therefore S_n &= \sin \alpha + \sin(\pi + 2\alpha) + \sin(2\pi + 3\alpha) + \\ &\quad \sin(3\pi + 4\alpha) + \dots + n \text{ terms} \\ &= \sin \alpha + \sin(\alpha + (\alpha + \pi)) + \sin(\alpha + (\alpha + 2\pi)) \\ &\quad + \dots + n \text{ terms} \end{aligned}$$

This is a sum of series of sines of angles which are in A.P. with first term and common difference $\alpha + \pi$.

Formula
$$S_n = \frac{\sin\left(\alpha + \frac{(n-1)\beta}{2}\right) \cdot \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Here $\alpha = \alpha$, $\beta = (\alpha + \pi)$.

$$S_n = \frac{\sin\left(\alpha + \frac{(n-1)(\alpha + \pi)}{2}\right) \sin\left(\frac{n(\alpha + \pi)}{2}\right)}{\sin\left(\frac{\alpha + \pi}{2}\right)}$$

$$= \frac{\sin\left(\alpha + \frac{(n-1)(\alpha + \pi)}{2}\right) \cdot \sin\left(\frac{n(\alpha + \pi)}{2}\right)}{\sin\left(\frac{\pi}{2} + \frac{\alpha}{2}\right)}$$

$$S_n = \frac{\sin\left(\alpha + \frac{(n-1)(\alpha + \pi)}{2}\right) \sin\left(\frac{n(\alpha + \pi)}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$$

1) Find $S_n = \cos^2 \alpha + \cos^2 (\alpha + \beta) + \cos^2 (\alpha + 2\beta) + \dots$ n terms

Hence find $\cos^2 \alpha + \cos^2 2\alpha + \cos^2 3\alpha + \dots$ n terms

Soln.

W.k.t $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

Given $S_n = \cos^2 \alpha + \cos^2 (\alpha + \beta) + \cos^2 (\alpha + 2\beta) + \dots$ n terms

$$= \left[\frac{1 + \cos 2\alpha}{2} \right] + \left[\frac{1 + \cos 2(\alpha + \beta)}{2} \right] +$$

$$\left[\frac{1 + \cos 2(\alpha + 2\beta)}{2} \right] + \dots$$
 n terms

$$= \frac{1}{2} + \frac{\cos 2\alpha}{2} + \frac{1}{2} + \frac{\cos 2(\alpha + \beta)}{2} + \frac{1}{2} +$$

$$\frac{\cos 2(\alpha + 2\beta)}{2} + \dots$$
 n terms

$$= \frac{n}{2} + \frac{1}{2} \left[\cos 2\alpha + \cos 2(\alpha + \beta) + \cos 2(\alpha + 2\beta) + \dots \right]$$
 n terms

Formula $S_n = \cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots$

$$S_n = \frac{\cos \left(\alpha + \frac{(n-1)\beta}{2} \right) \cdot \sin \left(\frac{n\beta}{2} \right)}{\sin \left(\frac{\beta}{2} \right)}$$

Here $\alpha = 2\alpha$ $\beta = 2\beta$

$$S_n = \frac{n}{2} + \frac{1}{2} \left[\frac{\cos \left(2\alpha + \frac{(n-1)2\beta}{2} \right) \cdot \sin \left(\frac{n2\beta}{2} \right)}{\sin \left(\frac{2\beta}{2} \right)} \right]$$

$$S_n = \frac{n}{2} + \frac{1}{2} \left[\frac{\cos (2\alpha + (n-1)\beta) \cdot \sin (n\beta)}{\sin \beta} \right]$$

Putting $\beta = \alpha$ in the above result

$$\cos^2 \alpha + \cos^2 2\alpha + \cos^2 3\alpha + \dots$$
 n terms
$$= \frac{n}{2} + \frac{1}{2} \left[\frac{\cos (2\alpha + (n-1)\alpha) \sin (n\alpha)}{\sin \alpha} \right]$$

$$= \frac{n}{2} + \frac{1}{2} \left[\frac{\cos (2\alpha + n\alpha - \alpha) \sin (n\alpha)}{\sin \alpha} \right]$$

$$= \frac{n}{2} + \frac{1}{2} \left[\frac{\cos(\alpha + n\alpha) \sin(n\alpha)}{\sin \alpha} \right]$$

$$S_n = \frac{n}{2} + \frac{1}{2} \left[\frac{\cos(n+1)\alpha \cdot \sin n\alpha}{\sin \alpha} \right]$$

5) Find $S_n = \cos \alpha \cos 3\alpha + \cos 3\alpha \cos 5\alpha + \cos 5\alpha \cos 7\alpha + \dots$ n terms.

Soln.

$$\text{Formula } \cos A \cdot \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$$

Given

$$S_n = \cos \alpha \cos 3\alpha + \cos 3\alpha \cos 5\alpha + \cos 5\alpha \cos 7\alpha + \dots + n \text{ terms}$$

$$= \frac{1}{2} [\cos(\alpha + 3\alpha) + \cos(3\alpha - \alpha)] + \frac{1}{2} [\cos(5\alpha + 3\alpha) + \cos(5\alpha - 3\alpha)] + \frac{1}{2} [\cos(7\alpha + 5\alpha) + \cos(7\alpha - 5\alpha)] + \dots n \text{ terms}$$

$$= \frac{1}{2} [\cos 4\alpha + \cos 2\alpha + \cos 8\alpha + \cos 2\alpha + \cos 12\alpha + \cos 2\alpha + \dots n \text{ terms}]$$

$$S_n = \frac{1}{2} [n \cos 2\alpha + \{\cos 4\alpha + \cos 8\alpha + \cos 12\alpha + \dots n \text{ terms}\}]$$

$$\text{Formula } S_n = \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots$$

$$S_n = \frac{\cos\left(\alpha + \frac{(n-1)\beta}{2}\right) \cdot \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

$$\text{Here } \alpha = 4\alpha \quad \beta = 4\alpha$$

$$S_n = \frac{1}{2} \left[n \cos 2\alpha + \frac{\cos\left(4\alpha + \frac{(n-1)4\alpha}{2}\right) \cdot \sin\left(\frac{n4\alpha}{2}\right)}{\sin\left(\frac{4\alpha}{2}\right)} \right]$$

$$= \frac{1}{2} \left[n \cos 2\alpha + \frac{\cos(4\alpha + 2n\alpha - 2\alpha) \cdot \sin(2n\alpha)}{\sin 2\alpha} \right]$$

$$S_n = \frac{1}{2} \left[n \cos 2\alpha + \frac{\cos(2\alpha + 2n\alpha) \cdot \sin 2n\alpha}{\sin 2\alpha} \right]$$

$$S_n = \frac{1}{2} \left[n \cos 2\alpha + \frac{\cos(2(n+1)\alpha) \sin 2\alpha}{\sin 2\alpha} \right]$$

b) Find $S_n = \cos^3 \alpha + \cos^3 2\alpha + \cos^3 3\alpha + \dots$ n terms

soln.

Formula $\cos 3A = 4\cos^3 A - 3\cos A$

Given $\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$

$$S_n = \cos^3 \alpha + \cos^3 2\alpha + \cos^3 3\alpha + \dots \text{ n terms}$$

$$= \frac{1}{4} [\cos 3\alpha + 3\cos \alpha] + \frac{1}{4} [\cos 3(2\alpha) + 3\cos 2\alpha] + \frac{1}{4} [\cos 3(3\alpha) + 3\cos 3\alpha] + \dots \text{ n terms}$$

$$= \frac{1}{4} [\cos 3\alpha + 3\cos \alpha] + \frac{1}{4} [\cos 6\alpha + 3\cos 2\alpha] + \frac{1}{4} [\cos 9\alpha + 3\cos 3\alpha] + \dots \text{ n terms}$$

$$= \frac{1}{4} \cos 3\alpha + \frac{3}{4} \cos \alpha + \frac{1}{4} \cos 6\alpha + \frac{3}{4} \cos 2\alpha + \frac{1}{4} \cos 9\alpha + \frac{3}{4} \cos 3\alpha + \dots \text{ n terms}$$

$$S_n = \frac{1}{4} [\cos 3\alpha + \cos 6\alpha + \cos 9\alpha + \dots \text{ n terms}] + \frac{3}{4} [\cos \alpha + \cos 2\alpha + \cos 3\alpha + \dots \text{ n terms}]$$

formula.

$$S_n = \frac{\cos\left(\alpha + \frac{(n-1)\beta}{2}\right) \cdot \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

$$S_n = \frac{1}{4} \left[\frac{\cos\left(3\alpha + \frac{(n-1)3\alpha}{2}\right) \cdot \sin\left(\frac{n3\alpha}{2}\right)}{\sin\left(\frac{3\alpha}{2}\right)} \right]$$

$$+ \frac{3}{4} \left[\frac{\cos\left(\alpha + \frac{(n-1)\alpha}{2}\right) \cdot \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \right]$$

$$= \frac{1}{4} \left[\frac{\cos\left(3\alpha + \frac{3n\alpha}{2} - \frac{3\alpha}{2}\right) \cdot \sin\left(\frac{3n\alpha}{2}\right)}{\sin\left(\frac{3\alpha}{2}\right)} \right]$$

$$+ \frac{3}{4} \left[\frac{\cos\left(\alpha + \frac{n\alpha}{2} - \frac{\alpha}{2}\right) \cdot \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \right]$$

$$= \frac{1}{4} \left[\frac{\cos \left(\frac{3\alpha}{2} + \frac{3n\alpha}{2} \right) \cdot \sin \left(\frac{3n\alpha}{2} \right)}{\sin \left(\frac{3\alpha}{2} \right)} \right] \\ + \frac{3}{4} \left[\frac{\cos \left(\frac{\alpha}{2} + \frac{n\alpha}{2} \right) \cdot \sin \left(\frac{n\alpha}{2} \right)}{\sin \left(\frac{\alpha}{2} \right)} \right]$$

$$S_n = \frac{1}{4} \left[\frac{\cos \left(\frac{3\alpha}{2} (n+1) \right) \cdot \sin \left(\frac{3n\alpha}{2} \right)}{\sin \left(\frac{3\alpha}{2} \right)} \right] + \\ \frac{3}{4} \left[\frac{\cos \left(\frac{\alpha}{2} (1+n) \right) \cdot \sin \left(\frac{n\alpha}{2} \right)}{\sin \left(\frac{\alpha}{2} \right)} \right]$$