Unit - I.

Transportation models and its variants

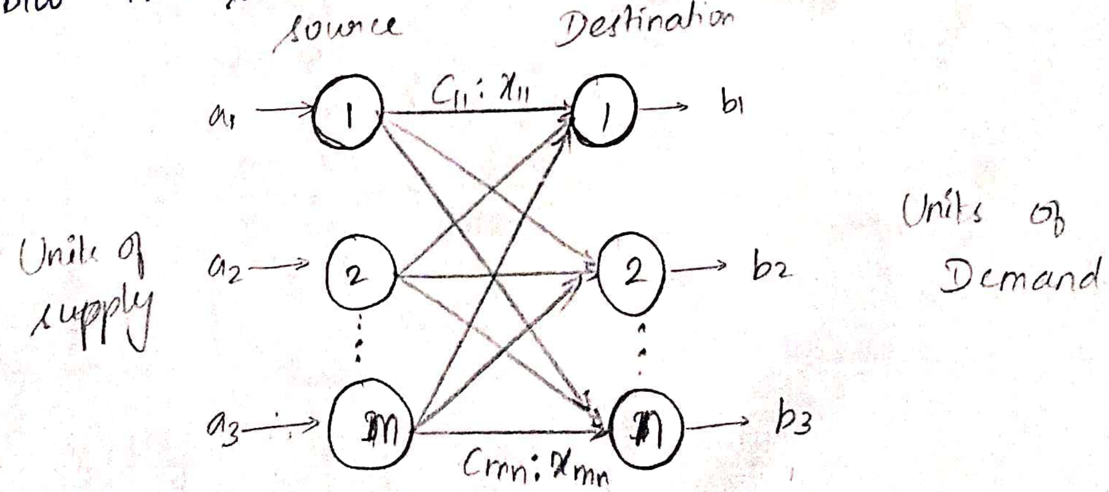
5.1 **Definition of the transportation Models:**

The transportation model is a special class of the linear programming problem. It deals with the situation in which a commodity is shipped from sources (eg: factories) to destinations (eg: warehouse)

The objective is to determine the amounts shipped from each sources to each destination that minimize the total shipping cost while satisfying both the supply limits and the demand requirements.

The model assumes that the shipping cost on a given route is directly proportional to the number of units shipped on the route.

The general problem is represented by the below network. There are m sources and n destinations each represented by a node. The arcs linking the sources and destinations represents the routes b/w the sources and destinations.



Arc (i,j) joining source i to destination j carries two pieces of information:

① the transportation cost per unit, c_{ij}.

② the amount shipped, x_{ij}.

→ The amount of supply at source i to a_i

→ The amount of demand at destination j to b_j.

The <u>objective</u> of the model is to determine the unknown x_{ij} that will <u>minimize</u> the total transportation cost while satisfying all the supply and demand restrictions.

Types of transportation:

There are 5 types of transportation.

i) NWC - North west corner rule.

ii) RM - Row Minima

iii) CM - Column minima

iv) LCM - Least cost method

v) VAM - Vogel's Approximation Method.

Destination

RR - Row Penalty

	W_1	W_2	W_3
F_1	x_{11} C_{11}	x_{12} C_{12}	x_{13} C_{13}
Source F_2	x_{21} C_{21}	x_{22} C_{22}	x_{23} C_{23}
F_3	x_{31} C_{31}	x_{32} C_{32}	x_{33} C_{33}

Availability (or) Supply

Requirement (or) Demand

CP - column Penalty

Example 5.1.1 MG Auto has three plants in Los Angeles, Detroit and New orleans, and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are 1000, 1500, and 1200 cars. The quarterly demand at the two distribution centers are 2300, 1400 cars. The mileage chart between the plants and the distribution centers are given in

	Denver	Miami	
LA	1000	2690	1000
Detroit	1250	1350	1500
New orleans	1275	850	1200
	2300	1400	

The trucking company in charge of transporting the cars charges 8 cents per mile per car. The transportation Then find the minimum transportation cost in dollar.

Solution:

Given the mileage chart b/w plants and distribution.

We have to convert it to dollars.

Also Given car charges 8 cents per mile per car

[∵ 1 dollar = 100 cen

i) 8 × 1000 = 8000 cents = 80 $

ii) 2690 × 8 = 21520 cents = 215.2 $

iii) 1250 × 8 = 10000 cents = 100 $

iv) 1350 × 8 = 108 $

v) 1275 × 8 = 102 $

vi) 850 × 8 = 68 $

(by NWC)

	Denver	Miami	
LA	[1000] 80	215.2	1000
Detroit	[1300] 100	[200] 108	1500
New orlean	102	[1200] 68	1200
	2300	1400	

∴ The optimal solution is obtained as follows:

shipping 1000 cars from LA to Denver

1300 cars from Detroit to Denver

200 cars from Detroit to Miami

1200 cars from New orlean to Miami

∴ The minimum transportation cost is

1000 x 80 $ = 80000
1300 x 100 $ = 130000
200 x 108 $ = 21600
1200 x 68 $ = 81600

$ 313200 → min cost

Algorithmas :

1) North west corner method :

Starts at North west corner cell (route) of the table.

Step ①:

Allocate as much as possible to the selected cell and adjust the associated amounts of supply and demand by subtracting the allocated amount

Step ②:

Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column.

If both the row and column net to zero simultaneously, cross out one only, and leave a zero supply (or) (demand) in the uncrossed out row or (column)

Step ③:

If exactly one row or column is left uncrossed out stop. otherwise, move to the cell to the right if a column has been crossed out or move to the cell to the one below if a row has been crossed out.

Go to step 1. until exactly one row or column is left uncrossed out

2) Least Cost Method : (LCM)

1) The least cost method finds a better starting solution by concentrating on the cheapest routes.

2) Instead of starting with the north west ~~corner~~ cell, we start by assigning as much as possible to the cell with the smallest unit cost.

3) Then we cross out the satisfied row (or) column and adjust the associated amounts of supply and demand accordingly.

4) If both a row or a column are satisfied simultaneously, only one is crossed out, the same as in NWC method. Next, we always look for the uncrossed out cell with the smallest unit cost.

5) Repeat the process until we are left at the end with exactly one uncrossed out row (or) column.

3) Vogel Approximation method (VAM):

Step ①: VAM is an improved version of the least cost method that generally produces better starting solutions.

Step ①:
For each row (each column) with strictly positive supply (demand), determine a penalty measure by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column)

Step ②:
Identify the row or column with the largest penalty. Break ties arbitrarily.
Allocate as much as possible to the variable with the smallest unit cost in the selected row or column.

Adjust the associated amount of supply and demand, and Cross out the satisfied row or column. If a row and column are satisfied simultaneously. only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step ③ :

a) If exactly one row or column with zero supply or demand remains uncrossed out, **stop**.

b) If one row (column) with positive supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least cost method, stop.

c) If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the zero basic variables by the least cost method. Stop.

d) otherwise go to step ①.

4) Hungarian method :

The assignment model is a special case of the transportation model where workers represent sources and jobs represent destinations.

The supply (demand) amount at each source (destination) exactly equals 1. The cost of transporting worker i to job j is c_{ij}.

Since all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the Hungarian method.

$$\begin{array}{cc|cccc|c} & & 1 & 2 & \cdots & N & \\ \hline & 1 & c_{11} & c_{12} & \cdots & c_{1n} & 1 \\ \text{workers} & 2 & c_{21} & c_{22} & \cdots & c_{2n} & 1 \\ & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & N & c_{n1} & c_{n2} & \cdots & c_{nn} & 1 \\ \hline \end{array}$$

Step ①:

For the original cost matrix, identify each row's minimum, and subtract it from all the entries of the rows.

Step ②:

For the matrix resulting from step 1, identify each column's minimum, and subtract it from all the entries of the column.

Step ③:

Identify the optimal assignment as the one associated with the zero elements of the matrix obtained in step ②.

In some cases, the zeros created by step ① and step ② may not yield a feasible solution directly. In this case, further steps are needed to find the optimal (feasible) assigment.

Step (2a):

If no feasible assignment (with all zero entries) can be secured from steps ① and ②.

i) Draw the minimum number of horizontal and verticals lines in the last reduced matrix that will cover all the zero entries.

ii) select the smallest uncovered element, and subtract it from every uncovered element; then add it to every element at the intersection of two lines.

iii) If no feasible assignment can be found among the optimal assignment, repeat step 2a. otherwise go to step ③ to determine the optimal assignment.

Example 5.1.2

In the MG model, suppose that the Detroit Plant capacity is 1300 cars (instead of 1500). Then the total supply = 3500 (cars) is less than the total demand = 3700 cars. Find the minimum cost.

Sol: Since supply is less than the demand, a dummy plant (source) with a capacity of 200 cars (=3700 - 3500) is added to balance the model.

The unit transportation cost from the dummy plant to the two destination is 0, because the plant does not exist.

	Denver	Miami	Supply
Los Angeles	1000 / 80	215	1000
Detroit	1300 / 100	108	1300
New orleans	102	1200 / 68	1200
Dummy plant	0	200 / 0	200
Demand	2300	1400	

∴ The optimal solution is obtained as follows:

Shipping 1000 cars from LA to Denver
1300 cars from Detroit to Denver
1200 cars from New orlean to Miami
200 cars from Dummy Plant to miami.
(means that miami will be 200 cars short of satisfying its demand of 1400 cars)

∴ The minimum cost of transportation is

1000 × \$80	= 80000
1300 × \$100	= 130000
1200 × \$68	= 81600
200 × \$0	= 0
	291600

Example 5.1.3 (refer Eg 5.1.1)

In the MG model, suppose that demand at Denver is 1900 cars instead of 2300 cars, find the min cost.

Soln: Since the demand is less than supply, a dummy plant (destination) with a capacity of 400 cars is added to balance the model.

The unit transportation cost from the dummy plant to the sources is 0, because the plant does not exist.

	Denver	Miami	Dummy plant	supply
LA	1000 / 80	215	0	1000
Detroit	900 / 100	200 / 108	400 / 0	1500
New orlean	102	1200 / 68	0	1200
demand	1900	1400	400	

∴ The optimal solution is obtained as follows:

shipping 1000 cars from LA to Denver
900 cars from Detroit to Denver
200 cars from Detroit to Miami
1200 cars from N.Orlean to Miami
400 cars from Detroit to Dummy plant.

∴ The minimum cost of transportation is

$$1000 \times \$80 = 80000$$
$$900 \times \$100 = 90000$$
$$200 \times \$108 = 21600$$
$$1200 \times \$68 = 81600$$
$$400 \times 0 = 0$$
$$\overline{273200}$$

Production - Inventory control:

Example 5.2.1

Boralis manufactures backpacks for serious hikers. The demand for its product during the peak period of March to June of each year is 100, 200, 180 and 300 units, respectively. The company uses part-time labor to accommodate fluctuations in demand. It is estimated that Boralis can produce 50, 180, 280, 270 units in March through June. A current month's demand may be satisfied in one of three ways.

i) current month's production at the cost of \$40 per pack.

2) Surplus production in an earlier month at an additional holding cost of \$.50 per pack per month.

3) Surplus production in a later month (back-ordering) at an additional penalty cost of ~~\$2.00~~ \$2.00 per pack per month.

Bonallu wishes to determine the optimal production ~~set~~ schedule for the four months.

Sol: The unit transportation cost from period i to period j is computed as,

$$c_{ij} = \begin{cases} \text{Production cost in } i, & \underline{i=j} \\ \text{Production cost in } i + \text{holding cost from } i \text{ to } j & \underline{i<j} \\ \text{Production cost in } i + \text{penalty cost from } i \text{ to } j & \underline{i>j} \end{cases}$$

For example; $c_{11} = \$40.00$

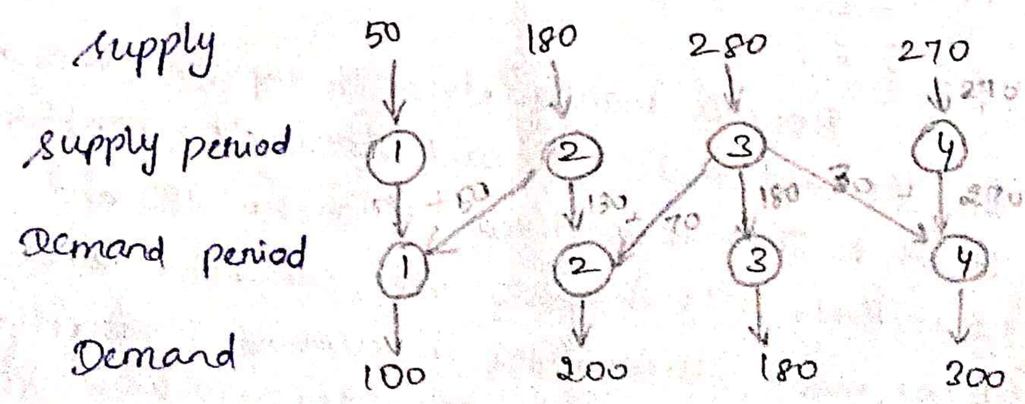
$c_{13} = \$40 + (\$0.5 + \$0.5) = \$41.00$

$c_{21} = \$40 + \$2.0 = \$42$.

	1	2	3	4	
1	50 \$40	\$40.5	\$41	\$41.5	50
2	~~50~~ \$42.	130 \$40	\$40.5	\$41	180 ~~130~~
3	\$44	70 \$42	180 \$40	30 \$40.5	280 ~~210~~ ~~30~~
4	\$46	\$44.	\$42	270 \$40	270
	100 ~~50~~	200	180	300 ~~270~~	

NWC

The optimal solution is summarized.



∴ The total cost is \$ 81,455

Example 5.2.2 Tool Sharpening (Equipment Maintenance)

Arkansas pacific operates a medium sized saw mill. The mill prepares different types of wood that range from soft pine to hard oak according to a weekly schedule. Depending on the type of wood being milled, the demand for sharp blades varies from day to day according to the following 1-week (7 day) data:

Day	Mon	Tues	Wed	Thur	Fri	Sat	Sun	
Demand (blades)	24	12	14	20	18	14	22	124

The mill can satisfy the daily demand in the following manner;

i) Buy new blades at the cost of $12 a blade.

2) Use an overnight sharpening service at the cost of $6 a blade, or a slow 2 day service at the cost of $3 a blade

Sol: The sources of the model are defined as follows:

* source 1 corresponds to buying new blades, (Provide supply to cover the demand for all 7 days).

* Source 2 to 8 corresponds to the 7 days of the week

The destinations represents the 7 days of the week.

The unit transportation cost for the model would be $12, $6, $3, depending on the blades is supplied from new blades, overnight sharpening or 2 day sharpening

	Mon	Tue	Wed	Thur	Fri	Sat	Sun	disposal	
New	24 12	12	12	12	12	12	12	100 0	124
Mon		12 6	2 6	10 3	3	3	3	0	24
Tues			12 6	6	3	3	3	0	12
Wed				10 6	4 6	3	3	0	14
Thur					14 6	6	6 3	0	20
Fri						14 6	4 6	0	18
Sat							12 6	2 0	14
Sun								22 0	22
	24	12	14	20	18	14	22	124	

The total cost is $840

Example 5.3.1 Sunray Transport

Sunray transport company ships truckloads of grain from three silos to four mills. The supply and the demand together with the unit transportation costs for truckload on the different route.

i) Northwest corner method

	1	2	3	4	
1	$\boxed{5}$ 10	$\boxed{10}$ 2	20	11	15
2	12	$\boxed{5}$ 7	$\boxed{15}$ 9	$\boxed{5}$ 20	25
3	4	14	16	$\boxed{10}$ 18	10
	5	15	15	15	

3 + 4 − 1 = 6 allotted cells

The method starts at NWC cell of table x_{11}

The starting basic solution is $x_{11} = 5$, $x_{12} = 10$, $x_{22} = 5$, $x_{23} = 15$, $x_{24} = 5$, $x_{34} = 10$.

The associated cost is

$5 \times 10 = 50$
$10 \times 2 = 20$
$5 \times 7 = 35$
$15 \times 9 = 135$
$5 \times 20 = 100$
$10 \times 18 = 180$

\$ 520

Example 5.3.2

Least cost method.

	1	2	3	4	
1	10	15 / 2	20	0 / 11	15
2	12	[illegible] / 7	15 / 9	10 / 20	25
3	5 / 4	14	16	5 / 18	10
	5	15	15	15	

The starting solution is $x_{12} = 15$, $x_{14} = 0$, $x_{23} = 15$, $x_{24} = 10$, $x_{31} = 5$, $x_{34} = 5$.

The associated cost is

$15 \times 2 = 30$
$0 \times 11 = 0$
$15 \times 9 = 135$
$10 \times 20 = 200$
$5 \times 4 = 20$
$5 \times 18 = 90$

\$ 475

Eg 5.3.4 Vogel's Approximation method.

	1	2	3	4		Row penalty		
1	10	15 / 2	20	11	15	8	9*	–
2	12	7	15 / 9	10 / 20	25	2	2	11*
3	5 / 4	0 / 14	16	5 / 18	~~10~~ 5	10*	2	2
	5	15	15	15				
Column penalty	6	5	7	7				
	–	5	7	7				
	–	–	7	2				

The starting solution is $x_{12} = 15$, $x_{23} = 15$, $x_{24} = 10$

$x_{31} = 5$, $x_{32} = 0$, $x_{34} = 5$.

The associated ~~solution~~ cost is

$15 \times 2 = 30$
$15 \times 9 = 135$
$10 \times 20 = 200$
$5 \times 4 = 20$
$0 \times 14 = 0$
$5 \times 18 = 90$

$\$475$

Eg 5.3.5 Solve the transportation model starting with NWC solution determine the solution using method of multipliers.

u-v method, modi method, optimed solution.

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$
$u_1 = 0$	5 / 10	10 / 2	20	11
$u_2 = 5$	12	5 / 7	15 / 9	5 / 20
$u_3 = 3$	4	14	16	10 / 18

$u_i + v_j - c_{ij}$ for alloted

We will arbitrarily set $u_1 = 0$ and solve the remaining variables.

Next we use u_i and v_j to evaluate the non-basic variables by computing $u_i + v_j - c_{ij}$. (determine all $u_i + v_j - c_{ij} \leq 0$)

Non basic variable	$u_i + v_j - c_{ij}$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 4 - 20 = -16 \leq 0$
x_{14}	$0 + 15 - 11 = 4 > 0$
x_{21}	$5 + 10 - 12 = 3 > 0$
x_{31}	$3 + 10 - 4 = 9 > 0$
x_{32}	$3 + 2 - 14 = -9 < 0$
x_{33}	$3 + 4 - 16 = -9 < 0$

The entering variable is the one having most +ve coefficient in the Z-row namely x_{31} is the entering variable.

Iteration ①

5 (θ−) / 10	10 (θ+) / 2	20	11
12	5 (θ−) / 7	15 / 9	5 (θ+) / 20
θ (+5) / 4	14	16	10 (θ−) / 18

min of θ (5, 5, 10)

min = 5 = θ

	$v_1=1$	$v_2=2$	$v_3=4$	$v_4=15$	
$u_1=0$	0 / 10	15 / 2	20	11	15
$u_2=5$	12	0 / 7	15 / 9	10 / 20	25
$u_3=3$	5 / 4	14	16	5 / 18	10
	5	15	15	15	

Non basic variable $\qquad\qquad$ $u_i+v_j-c_{ij}$

x_{11} $\quad 0+1-10=-9$

x_{13} $\quad 0+4-20=-16$

x_{14} $\quad 0+15-11=4>0$

x_{21} $\quad 5+1-12=-6$

x_{32} $\quad 3+2-14=-9$

x_{33} $\quad 3+4-16=-9$

Iteration ②

0 / 10	15 (θ−) / 2	20	θ / 11
12	0 (θ+) / 7	15 / 9	10 (θ−) / 20
5 / 4	14	16	5 / 18

min of θ (10, 15)

min = 10 = θ

	$v_1=3$	$v_2=2$	$v_3=4$	$v_4=11$	
$u_1=0$	10	5 / 2	20	10 / 11	15
$u_2=5$	12	10 / 7	15 / 9	20	25
$u_3=7$	5 / 4	14	16	5 / 18	10
	5	15	15	15	

Non basic variable $\qquad\qquad$ $u_i+v_j-c_{ij}$

x_{11} $\quad 0+3-10=-13$

x_{13} $\quad 0+4-20=-16$

x_{21} $\quad 5-3-12=-10$

x_{24} $\quad 5+11-20=-4$

x_{32} $\quad 7+2-14=-5$

x_{33} $\quad 7+4-16=-5$

<0

Optimum solution.

From silo	To mill	No of truckload		
1	2	5	× 2	= 10
1	4	10	× 11	= 110
2	2	10	× 7	= 70
2	3	15	× 9	= 135
3	1	5	× 4	= 20
3	4	5	× 18	= 90
				$435

Assignment Problem

Hungarian method:

Example 5.4.1

Joe Klyne's three children, John, Karen and Terri, want to earn some money for personal expenses. Mr. Klyne has chosen three chores for his children: mowing the lawn, painting the garage door and washing the family cars. To avoid anticipated sibling competition he asks them to submit individual bids for what they feel is fair pay for each of the three chores, solve the assignment problem by the Hungarian method.

Sol

	Mow	paint	wash
John	15	10	9
Karen	9	15	10
Terri	10	12	8

Calculate row minimum and subtract it from all the elements and then calculate the column minimum.

	Mow	paint	wash	Row min
John	15	10	9	$p_1 = 9$
Karen	9	15	10	$p_2 = 9$
Terri	10	12	8	$p_3 = 8$

⇒

	Mow	paint	wash
J	6	1	0
K	0	6	1
T	2	4	0
	$q_1 = 0$	$q_2 = 1$	$q_3 = 0$

⇒

	Mow	paint	wash
J	6 [15]	0 [10]	0 [9]
K	0 [9]	5 [15]	1 [10]
T	2 [10]	3 [12]	0 [8]

lines are equal to row and column

Step

∴ The total cost to Mr. Klyne is 9 + 10 + 8 = $27

This amount also will equal to $(p_1 + p_2 + p_3) + (q_1 + q_2 + q_3)$

$\Rightarrow (9 + 9 + 8) + (0 + 1 + 0)$

$= \$27$

Example 15.4.2

Suppose that the situation discussed in Eg 5.4.1 is extended to four children and four chores.

Chores

Child	1	2	3	4	Row min
1	1	4	6	3	1
2	9	7	10	9	7
3	4	5	11	7	4
4	8	7	8	5	5

Calculate the row minimum and subtract it from all the elements in that row and then calculate the column minimum.

	1	2	3	4
1	0	3	5	2
2	2	0	3	2
3	0	1	7	3
4	3	2	3	0
Column min	0	0	3	0

⟹

	1	2	3	4
1	0	3	2	2
2	2	0	0	2
3	0	1	4	3
4	3	2	0	0

Since the lines ~~are not equal~~ does not cover all the zeros to row and column. Then choose ~~minimum~~ smallest in uncovered entry of non striking lines

~~Minus 1 in uncovered entry~~, subtract it from every uncovered entry and then add it to every entry at the intersection of two lines.

	1	2	3	4
1	0	2	1	1
2	3	0	0	2
3	0	0	3	2
4	4	2	0	0

∴ The total cost is

1+5+10+5

= \$21

Draw the minimum no of horizontal or vertical lines in the last reduced matrix to cover all the 0 entries.