Third part of Sylow's theorem

The number of p-sylow subgroups in G for a given prime is of the form $1+kp$.

Proof:

Let P be a p-sylow subgroup of G s.t. $o(P)=p^n$.

We decompose G into double cosets of P and P

thus $G = \cup PxP$.

$$o(G) = \sum_{x \in G} o(PxP)$$, where the summation

taken over distinct double cosets.

$$o(G) = \sum_{x \in N(P)} o(PxP) + \sum_{x \notin N(P)} o(PxP) \text{ ---}$$

For $x \in N(P)$,

Consider $PxP = PPx = Px$ ---(*)

From ①, $o(G) = \sum_{x \in N(P)} o(PxP) + \sum_{x \notin N(P)} o(PxP) \rightarrow ②$

For, $x \notin N(P)$, consider PxP

$Px \neq xP \ (\because x \notin N(P))$

$\Rightarrow Pxx^{-1} \neq xPx^{-1}$

(ie) $xPx^{-1} \neq P$

$\Rightarrow P \cap xPx^{-1} \neq P \cap P$

$\Rightarrow P \cap xPx^{-1} \neq P$

$\therefore P \cap xPx^{-1}$ is a proper subgroup of P.

$o(P \cap xPx^{-1}) = p^m$, where $m < n$.

$$o(PxP) = \frac{o(P)\,o(P)}{o(P \cap xPx^{-1})}$$

$$= \frac{p^n p^n}{p^m}$$

$$= p^{2n-m} \geq p^{n+1} \quad (\because 2n-m \geq n+1)$$

$\Rightarrow o(PxP) \geq p^{n+1}$

$\Rightarrow p^{n+1} \mid o(PxP)$

$\Rightarrow p^{n+1} \mid \sum_{x \notin N(P)} o(PxP)$

$$\sum_{x \notin N(P)} o(PxP) = up^{n+1}, \text{ where } u \text{ is an integer.} \quad (3)$$

$$(2) \Rightarrow o(G) = \sum_{x \in N(P)} o(Px) + up^{n+1} \text{ (by (3))}.$$

$$o(G) = o(N(P)) + up^{n+1}.$$

$$\frac{o(G)}{o(N(P))} = \frac{o[N(P)]}{o(N(P))} + \frac{up^{n+1}}{o(N(P))} \quad \text{——— (4)}$$

$\dfrac{o(G)}{o(N(P))}$ is an integer which is the number of p-sylow subgroups.

Also $\dfrac{up^{n+1}}{o(N(P))}$ is an integer.

Since $p^{n+1} \nmid o(G) \Rightarrow p^{n+1} \nmid o(N(P))$.

$$\Rightarrow p \mid \frac{up^{n+1}}{o(N(P))}$$

$$\Rightarrow \frac{up^{n+1}}{o(N(P))} = kp.$$

$$(4) \Rightarrow \frac{o(G)}{o(N(P))} = 1 + kp.$$

(ie) the number of p-sylow subgroup is $1+kp$.