Unit - I

Linear Equations with constant coefficients:-

Introduction: Linear Differential eqn

* A linear differential eqn of order n with constant coeffs is an eqn of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = b(x),$$

where $a_0 \neq 0$, $a_1, a_2, \dots, a_n$ are complex constants and b is some complex valued fun on an interval I.

* By dividing by a_0 we can arrive at an eqn of the same form with a_0 replaced by 1.

$\therefore$ we can always assume $a_0 = 1$, and our eqn becomes

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = b(x) \quad \text{--- (1)}$$

Notation:

$$\therefore L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y \cancel{= b(x)}$$

and the eqn (1) becomes simply $L(y) = b(x)$.

Defn:

* A linear differential eqn of order n, $L(y) = b(x)$ is called a homogeneous eqn if $b(x) = 0$ $\forall x$ in I.
* A linear differential eqn of order n, $L(y) = b(x)$ is called a non homogeneous eqn if $b(x) \neq 0$ for some x in I.

* We give a meaning to L itself as a differential operator which operates on fun^s which have n derivatives on I, and transforms such a fun φ into a fun $L(\varphi)$ whose value at x is gn by

$$L(\varphi)(x) = \varphi^{(n)}(x) + a_1 \varphi^{(n-1)}(x) + \dots + a_n \varphi(x).$$

$$L(\varphi) = \varphi^{(n)} + a_1 \varphi^{(n-1)} + \dots + a_n \varphi.$$

ie) φ is a soln of $L(y) = b(x)$, then $L(\varphi) = b$

~~Then~~ If b is cts on I, then it is possible to find all solns of $L(y) = b(x)$.

2. Second order homogeneous eqns.

* The general eqn is $L(y) = y'' + a_1 y' + a_2 y = 0$. — ②

where a_1 and a_2 are constants.

* The first order eqn with constant coeffs $y' + ay = 0$ has a soln e^{-ax}. The constant $-a$ in this soln is the soln of the eqn $r + a = 0$.

* Since differentiating an exponential e^{rx} any no. times, where r is a constant, always yields a constant times e^{rx}, it is reasonable to expect that for some appropriate constant r, e^{rx} will be a soln of the eqn ②.

(not need)

The characteristic poly. of eqn ② is

$p(r) = r^2 + a_1 r + a_2$.

Solving $p(r) = 0$, we get the two roots namely r_1 and r_2. Then the soln. is:

$\varphi = c_1 \varphi_1 + c_2 \varphi_2$ where c_1 and c_2 are arbitrary constants.

* If $r_1 \neq r_2 \Rightarrow \varphi_1 = e^{r_1 x}, \quad \varphi_2 = e^{r_2 x}$

$\Rightarrow \varphi = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

* If $r_1 = r_2 \Rightarrow \varphi_1 = e^{r_1 x}, \quad \varphi_2 = x e^{r_1 x}$

$\Rightarrow \varphi = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$.

Let us try it for ②, We formulate the result as a thm.

Theorem 1.1: Let a_1, a_2 be constants and consider the eqn $L(y) = y'' + a_1 y' + a_2 y = 0$. (i) If r_1, r_2 are distinct roots of the characteristic polynomial p, where $p(r) = r^2 + a_1 r + a_2$, then the fns φ_1, φ_2 defined by $\varphi_1(x) = e^{r_1 x}, \quad \varphi_2(x) = e^{r_2 x}$ ③ are solns of $L(y) = 0$.

(ii) If r_1 is a repeated root of the p, then the fns φ_1, φ_2 defined by $\varphi_1 = e^{r_1 x}, \quad \varphi_2 = x e^{r_1 x}$ ④ are solns of $L(y) = 0$

Proof:

Consider the eqn $L(y) = y'' + a_1 y' + a_2 y = 0$ where a_1, a_2 are constants.

Now, Consider the fun e^{rx}. Then we find that

$$L(e^{rx}) = (e^{rx})'' + a_1 (e^{rx})' + a_2 (e^{rx})$$
$$= r(e^{rx})' + a_1 r e^{rx} + a_2 e^{rx}$$
$$= r^2 e^{rx} + a_1 r e^{rx} + a_2 e^{rx}$$
$$= (r^2 + a_1 r + a_2) e^{rx},$$

and e^{rx} will be a soln. of $L(y) = 0$.

ie) $L(e^{rx}) = 0$, if r satisfies $r^2 + a_1 r + a_2 = 0$.

we let $p(r) = r^2 + a_1 r + a_2$, and call p the characteristic polynomial of L, or of the eqn (2).

Note: $p(r)$ can be obtained from $L(y)$ replacing $y^{(k)}$ everywhere by r^k, where we use the conventions that the zeroth derivative of y, $y^{(0)}$, is y itself, and that $r^0 = 1$.

From the Fundamental Theorem of Algebra, we know that the polynomial p always has two complex roots r_1, r_2 (which may be real).

* If $r_1 \neq r_2$, then $p(r_1) = 0 = p(r_2)$.

$\therefore$ $L(e^{r_1 x}) = 0 = L(e^{r_2 x})$

Hence $e^{r_1 x}$ and $e^{r_2 x}$ are two distinct solns of $L(y)=0$.
It is possible to find two distinct solns in the case $r_1 = r_2$ also.

Since r_1 is a soln. root of $P(r)$, $P(r_1)=0$. Hence $e^{r_1 x}$ is one soln of $L(y)=0$. Also we have

$$L(e^{rx}) = p(r)e^{rx} \quad \text{— ⑤}$$

for all r and x.

If r_1 is a repeated root of p, then not only $p(r_1)=0$, but $p'(r_1)=0$. This suggests differentiating the eqn ⑤ w.r.to r. Then we observe that, since L involves only differentiation w.r.to x,

$$\frac{\partial}{\partial r} L(e^{rx}) = L\left(\frac{\partial}{\partial r} e^{rx}\right) = L(xe^{rx})$$

and therefore

$$L(xe^{rx}) = (xe^{rx})'' + a_1(xe^{rx})' + a_2(xe^{rx})$$
$$= (xre^{rx} + e^{rx})' + a_1(xre^{rx} + e^{rx}) + a_2 xe^{rx}$$
$$= (xr^2e^{rx} + re^{rx} + re^{rx}) + a_1e^{rx} + a_1xre^{rx} + a_2xe^{rx}$$
$$= xr^2e^{rx} + 2re^{rx} + a_1e^{rx} + a_1xre^{rx} + a_2xe^{rx}$$
$$= (xr^2 + 2r + a_1 + a_1xr + a_2x)e^{rx}$$
$$= (p'(r) + xp(r))e^{rx}$$

$\left(\because p(r) = r^2 + a_1 r + a_2 \right.$
$p'(r) = 2r + a_1$

Now setting $r = r_1$ in this eqn we see that $L(xe^{r_1 x}) = 0$, thus showing that $xe^{r_1 x}$ is another soln in case $r_1 = r_2$. Hence the theorem.

Remark.

If ϕ_1, ϕ_2 are any two solns. of $L(y)=0$, c_1, c_2 are any two constants, then the linear combination of two solns. $\phi = c_1\phi_1 + c_2\phi_2$ is also a soln. of the eqn. $L(y)=0$.

Indeed $L(\phi) = (c_1\phi_1 + c_2\phi_2)'' + a_1(c_1\phi_1 + c_2\phi_2)' + a_2(c_1\phi_1 + c_2\phi_2)$

$= c_1\phi_1'' + c_2\phi_2'' + c_1a_1\phi_1' + a_1c_2\phi_2' + c_1a_2\phi_1 + c_2a_2\phi_2$

$= \cancel{c_1 L(\phi_1)}$

$= c_1\phi_1'' + c_1a_1\phi_1' + c_1a_2\phi_1 + c_2\phi_2'' + c_2a_1\phi_2' + c_2a_2\phi_2$

$= c_1(\phi_1'' + a_1\phi_1' + a_2\phi) + c_2(\phi_2'' + a_1\phi_2' + a_2\phi)$

$= c_1 L(\phi_1) + c_2 L(\phi_2) = 0$

The fun. ϕ which is zero for all x is also a soln., the trivial soln. of $L(y)=0$.

Example 1.3 Consider the eqn. $\phi = y'' + y' - 2y = 0$.

The characteristic poly. is $p(r) = r^2 + r - 2 = 0$

$(r-1)(r+2) = 0$

$r = 1, -2$

$\therefore \phi_1 = e^{x}$ and $\phi_2 = e^{-2x}$.

Then the solⁿ is $\phi = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

i.e) $\phi = c_1 e^{x} + c_2 e^{-2x}$.

Eg: 2: Solve $y'' + \omega^2 y = 0$, where ω is a +ve Constant.

Solⁿ: The characteristic poly. is

$$P(r) = r^2 + \omega^2 = 0$$

$$r^2 = -\omega^2 \Rightarrow r = \pm i\omega$$

$$\therefore \varphi_1 = e^{i\omega x}, \quad \varphi_2 = e^{-i\omega x}$$

Then the solⁿ is $\phi = c_1 e^{i\omega x} + c_2 e^{-i\omega x}$ where c_1, c_2 are constants.

Taking $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$ we see that

$$\varphi = \tfrac{1}{2} e^{i\omega x} + \tfrac{1}{2} e^{-i\omega x}$$

$$= \frac{e^{i\omega x} + e^{-i\omega x}}{2}$$

$$\varphi = \cos\omega x.$$

$\therefore \cos\omega x$ is a solⁿ.

IIIly, taking $c_1 = \frac{1}{2i}$, $c_2 = \frac{-1}{2i}$ we see that

$$\varphi = \tfrac{1}{2i} e^{i\omega x} - \tfrac{1}{2i} e^{-i\omega x}$$

$$= \frac{e^{i\omega x} - e^{-i\omega x}}{2i}$$

harmonic oscillator is modeled by a linear second order ODE

$\varphi(x) = \sin\omega x$.

$\therefore$ $\sin\omega x$ is a soln. The eqn $y'' + \omega^2 y = 0$ is called the harmonic oscillator eqn.

Exercises:

1) Solve $y'' - 4y = 0$

Soln Let $L(y) = y'' - 4y$

The characteristic eqn. is

$p(r) = r^2 - 4 = 0$

$\Rightarrow r^2 = 4 \Rightarrow r = \pm 2 \Rightarrow r_1 = 2, \quad r_2 = -2$

$\therefore \varphi_1 = e^{2x} \qquad \varphi_2 = e^{-2x}$

Then the soln. is $\varphi = c_1 e^{2x} + c_2 e^{-2x}$

2) Solve $3y'' + 2y' = 0$

Soln. Let $L(y) = 3y'' + 2y'$

The characteristic eqn is $p(r) = 3r^2 + 2r$

$3r^2 + 2r = 0$

$r(3r+2) = 0 \Rightarrow r_1 = 0 \quad r_2 = -2/3$

$\therefore \varphi_1 = e^{r_1 x} = e^{0x} = 1 \qquad \varphi_2 = e^{-2/3 x}$

Then the soln. is $\varphi = c_1 + c_2 e^{-2/3 x}$

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3. Solve $y'' + y' - 6y = 0$.

Soln:

Let $L(y) = y'' + y' - 6y$

The characteristic eqn is $P(r) = r^2 + r - 6y = 0$

$$(r+3)(r-2) = 0$$

$$r_2 = -3 \quad r_1 = 2$$

$$\therefore \phi_1 = e^{r_2 x} = e^{-3x}, \quad \phi_2 = e^{r_1 x} = e^{2x}$$

Then the soln. is $\phi = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

$$\phi = c_1 e^{+2x} + c_2 e^{-3x}$$

4. $y'' + (3i-1)y' - 3iy = 0$.

Soln:

Let $L(y) = y'' + (3i-1)y' - 3iy = 0$

The characteristic eqn is

$$P(r) = r^2 + (3i-1)r - 3i = 0$$

$$r = \frac{-(3i-1) \pm \sqrt{(3i-1)^2 - 4(-3i)}}{2(1)}$$

$$= \frac{-3i+1 \pm \sqrt{-9+1-6i+12i}}{2}$$

$$= \frac{-3i+1 \pm \sqrt{-8+6i}}{2}$$

$\sqrt{-8+6i} = x+iy$

Squaring on both sides

$-8+6i = (x+iy)^2$

$-8+6i = x^2 - y^2 + 2i$

$\boxed{x^2 - y^2 = -8}$

$2xy = 6$

$\boxed{xy = 3}$

solving these eqn.

$\boxed{x = \frac{9}{3}}$

$\frac{y^2}{x} - y^2 = -8$

$y^2 - 3y^2 = -24$

$-2y^2 = -24$

$y^2 = 12$

$x^2 - 12 = -8$

$x^2 = 20$

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(or)

$y'' + (3i-1)y' - 3iy = 0$

soln

Let $L(y) = y'' + (3i-1)y' - 3iy = 0$

The characteristic eqn is

$$p(r) = r^2 + (3i-1)r - 3i = 0$$

$$\Rightarrow r^2 + 3ir - r - 3i = 0$$

$$r^2 - r + 3ir - 3i = 0$$

$$r(r-1) + 3i(r-1) = 0$$

$$(r+3i)(r-1) = 0$$

$$r = -3i \quad r = 1.$$

$\therefore \phi_2(x) = e^{-3ix} \qquad \phi_1(x) = e^{x}$

$\therefore$ Then the soln. is $\phi = c_2 e^{-3ix} + c_1 e^{x}$

1.3 Intial value Problem.

Every soln. of the eqn.

$L(y) = y'' + a_1 y' + a_2 y = 0$ is a linear combination of the solutions $\varphi_1(x) = e^{r_1 x}$, $\varphi_2(x) = e^{r_2 x}$ (or $\varphi_1(x) = e^{r_1 x}$, $\varphi_2(x) = x e^{r_1 x}$ will depend on showing that the initial value problems for this equation have unique solutions. An initial value problem for $L(y)=0$ is a problem of finding a soln. φ satisfying

starting position ← $\varphi(x_0) = \alpha$, $\varphi'(x_0) = \beta$ → starting speed / slope ——— (1.6)

where x_0 is some real no, and α, β are two given constants. Thus we specify φ and its first derivative at some initial point x_0. This problem is denoted by

$$L(y) = 0, \quad y(x_0) = \alpha, \quad y'(x_0) = \beta \qquad \text{——— (1.7)}$$

Theorem 2: (Existence Theorem)

For any real x_0, and constants α, β, there exists a soln. φ of the initial value problem $L(y) = 0$, $y(x_0) = \alpha$, $y'(x_0) = \beta$ on $-\infty < x < \infty$

(this thm. said there is exactly one specific path

(unique constants c_1 & c_2).

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Proof:

To prove: There exist a unique constants c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$ satisfying initial value problem.

By theorem 1.1, there exist a soln.

$\phi = c_1\phi_1 + c_2\phi_2$ for $L(y) = 0$.

At x_0, $\phi(x_0) = \alpha \Rightarrow c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha$

$\phi'(x_0) = \beta \Rightarrow c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta$ — (*)

By representing the eqns (*) in the matrix form, we have

$\begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{vmatrix}$

and ~~these eqn~~ this matrix will have a unique soln. $\therefore c_1, c_2$ if the determinant,

(In algebra, a system of 2 equations has a unique soln iff determinant $\neq 0$).

$$\Delta = \begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{vmatrix}$$

$$= \phi_1(x_0)\phi_2'(x_0) - \phi_1'(x_0)\phi_2(x_0) \quad \text{— (1)}$$

$$\neq 0$$

Case (i): If $r_1 \neq r_2$

$\phi_1(x_0) = e^{r_1 x_0}$, $\phi_2(x_0) = e^{r_2 x_0}$

$\phi_1'(x_0) = r_1 e^{r_1 x_0}$, $\phi_2'(x_0) = r_2 e^{r_2 x_0}$

$\therefore (1) \Rightarrow \phi_1(x_0)\phi_2'(x_0) - \phi_1'(x_0)\phi_2(x_0) = e^{r_1 x_0} \cdot r_2 e^{r_2 x_0} - e^{r_2 x_0} r_1 e^{r_1 x_0}$ ~~$\neq 0$~~

$= e^{r_1 x_0} e^{r_2 x_0} (r_2 - r_1)$ ~~$\neq 0$~~.

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$$(r_2 - r_1)\, e^{(r_1+r_2)x_0} \neq 0$$

which is not zero, since $e^{(r_1+r_2)x_0} \neq 0$ and $r_1 \neq r_2$.

Case (ii) $r_1 = r_2$

$\phi_1(x_0) = e^{r_1 x_0}$ $\qquad$ $\phi_2(x_0) = x_0 e^{r_1 (x_0)}$

$\phi_1'(x_0) = r_1 e^{r_1 x_0}$ $\qquad$ $\phi_2'(x_0) = x_0 r_1 e^{r_1 x_0} + e^{r_1 x_0}$

$$\therefore (1) \Rightarrow \Delta = e^{r_1 x_0} \cdot (x_0 r_1 e^{r_1 x_0} + e^{r_1 x_0}) - r_1 e^{r_1 x_0} \cdot x_0 e^{r_1 x_0}$$

$$= \cancel{x_0 r_1 e^{2r_1 x_0}} + e^{2r_1 x_0} - \cancel{x_0 r_1} e^{r_1 x_0}$$

$$= e^{2r_1 x_0} \neq 0.$$

$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

Therefore the determinant condition is satisfied in either case. Thus, if c_1, c_2 are the unique constants satisfying eqn. (*), the fun.

$\phi = c_1\phi_1 + c_2\phi_2$ will be the desired soln. satisfying $\phi(x_0) = \alpha$, $\phi'(x_0) = \beta$.

Remark:

We have shown that there is a unique linear combination of ϕ_1 and ϕ_2 which is a soln. of $L(y) = 0$, $\phi(x_0) = \alpha$, $\phi'(x_0) = \beta$. Although it is not quite obvious, it turns out that this soln. is the only one. Before proving this we

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give an estimate for the rate growth of any soln ϕ of $L(y)=0$, and its first derivative ϕ', in terms of the coeffs: $1, a_1, a_2$ appearing in $L(y)$. As a measure of the size of ϕ and ϕ' we take

$$\|\phi(x)\| = [|\phi(x)|^2 + |\phi'(x)|^2]^{1/2}.$$

where the positive square root is understood.
The "size" of L will be measured by

$$K = 1 + |a_1| + |a_2|.$$

Note: If b and c are any two constants, then we have the inequality that

$$2|b||c| \leq |b|^2 + |c|^2$$

This inequality results by noticing that

$$0 \leq (|b| - |c|)^2 = |b|^2 + |c|^2 - 2|b||c|.$$

Theorem: 3 Let ϕ be any soln. of $L(y)$
$L(y) = y'' + a_1 y' + a_2 y = 0$ on an interval I containing a point x_0. Then for all x in I

$$\|\phi(x_0)\| e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{k|x-x_0|}$$

where $\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2\right]^{1/2}$,

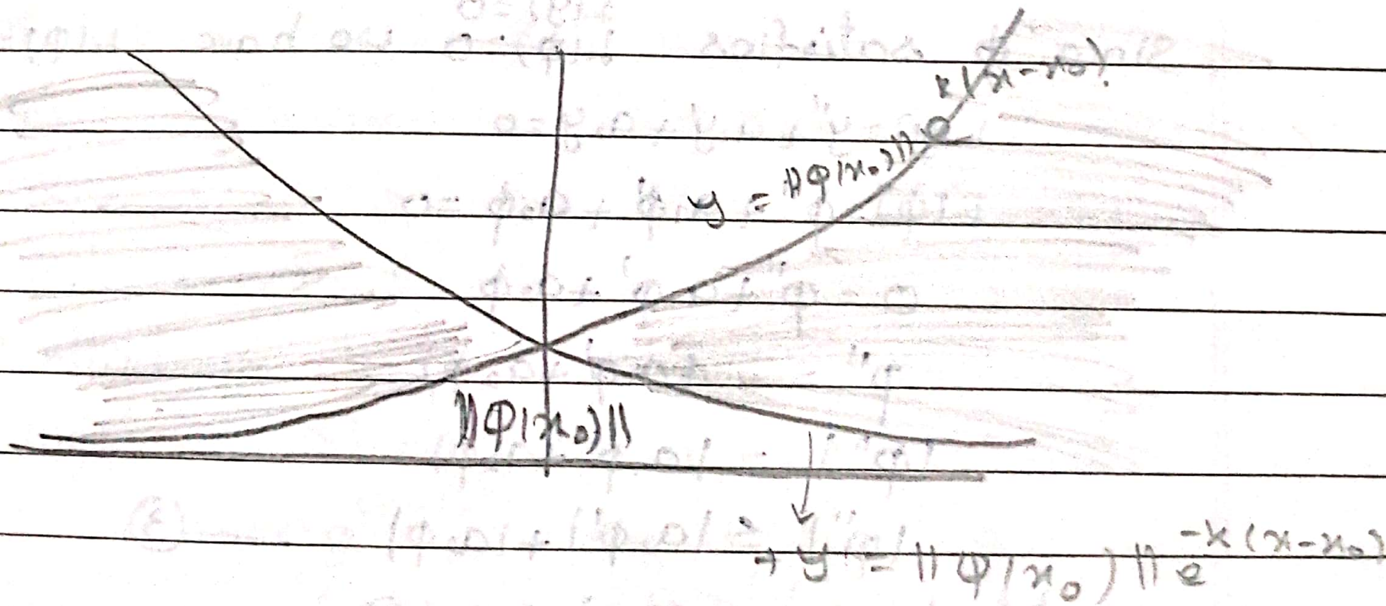
$k = 1 + |a_1| + |a_2|$.

Proof:

Remark: Geometrically the inequality ① says that $\|\phi(x)\|$ always remains b/t the two curves.

$$y = \|\phi(x_0)\| e^{-k(x-x_0)} \quad \text{and} \quad y = \|\phi(x_0)\| e^{k(x-x_0)}$$

as shaded area



Proof of theorem:

We let $u(x) = \|\phi(x)\|^2$ for $x \in I$. Then

$$u(x) = |\phi(x)|^2 + |\phi'(x)|^2$$

$$= \phi(x)\overline{\phi(x)} + \phi'(x)\overline{\phi'(x)} \qquad (\because |z|^2 = z\bar{z})$$

$$u(x) = \phi\bar{\phi} + \phi'\overline{\phi'}$$

where $\bar{\phi}(x) = \overline{\phi(x)}$, $\overline{\phi'(x)} = \bar{\phi}'(x)$.

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Then $u' = \phi'\bar{\phi} + \phi\bar{\phi'} + \phi''\bar{\phi'} + \phi'\bar{\phi''}$

$|u'| = |\phi'\bar{\phi} + \phi\bar{\phi'} + \phi''\bar{\phi'} + \phi'\bar{\phi''}|$

$\leq |\phi'\bar{\phi}| + |\phi\bar{\phi'}| + |\phi''\bar{\phi'}| + |\phi'\bar{\phi''}|$

$= |\phi'||\bar{\phi}| + |\phi||\bar{\phi'}| + |\phi''||\bar{\phi'}| + |\phi'||\bar{\phi''}|$

$= |\phi'||\phi| + |\phi||\phi'| + |\phi''||\phi'| + |\phi'||\phi''|$

$= 2|\phi||\phi'| + 2|\phi'||\phi''|$

~~$|u'| \leq 2|$~~

$|u'(x)| \leq 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||\phi''(x)|$ —— (2)

Since ϕ satisfies $L(y)=0$ ~~$L(\phi)=0$~~ we have $L(\phi)=0$

$\therefore L(y) = y'' + a_1 y' + a_2 y = 0$

$L(\phi) = \phi'' + a_1\phi' + a_2\phi = 0$

$0 = \phi'' + a_1\phi' + a_2\phi$

$\phi'' = -(a_1\phi' + a_2\phi)$

$|\phi''| = |a_1\phi' + a_2\phi|$

$|\phi''| \leq |a_1\phi'| + |a_2\phi|$ —— (3)

substitude eqn (3) in eqn (2)

$\therefore |u'(x)| \leq 2|\phi(x)||\phi'(x)| + 2|\phi'(x)|[|a_1\phi'(x)| + |a_2\phi(x)|]$

$= 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||a_1||\phi'(x)| + 2|\phi'(x)||a_2||\phi(x)|$

$= 2(1+|a_2|)|\phi(x)||\phi'(x)| + 2|a_1||\phi'(x)|^2$

$1+2|a_1|-2|a_1|$ $\quad 2|a_1||\phi(x)||\phi'(x)| \le |a_1|\left(|\phi(x)|^2+|\phi'(x)|^2\right)$

$1+|a_2|\,|\phi(x)|^2+|\phi'(x)|^2+|a_1|\left(|\phi(x)|^2+|\phi'(x)|^2\right)$

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$(1+|a_2|)\,|\phi(x)|^2+|\phi'(x)|^2+2|a_1|\,|\phi'(x)|^2$

$\left[\because\ 2|b||c| \le |b|^2+|c|^2\right]$ —— (*)

$$\le (1+|a_2|)\left(|\phi(x)|^2+|\phi'(x)|^2\right)+2|a_1||\phi'(x)|^2; \ (\text{by } (*))$$

$$= (1+|a_2|)\,(|\phi(x)|^2+(1+|a_2|)|\phi'(x)|^2+ 2|a_1||\phi'(x)|^2$$

$$= (1+|a_2|)|\phi(x)|^2+(1+|a_2|+2|a_1|)\,|\phi'(x)|^2$$

$$= (1+|a_2|)\,|\phi(x)|^2+(1+|a_2|)|\phi'(x)|^2+2|a_1|\,\phi'(x)^2$$

$$\le 2(1+|a_1|+|a_2|)|\phi(x)|^2+(1+|a_1|+|a_2|)\,|\phi'(x)|^2$$

$$= 2(1+|a_1|+|a_2|)\left(|\phi(x)|^2+|\phi'(x)|^2\right)$$

$$= 2\,k\,u(x), \text{ where } k = 1+|a_1|+|a_2|.$$

Therefore $|u'(x)| \le 2k\,u(x)$.

i.e. $\quad -2ku(x) \le u'(x) \le 2ku(x)$ —— (4)

Take $u'(x) \le 2ku(x)$. Then

$$u'(x)-2ku(x) \le 0$$

Multiplying by $e^{-2ku(x)}$ on both sides

$$e^{-2ku(x)}\left(u'(x)-2ku(x)\right) \le 0$$

$$u'(x)e^{-2ku(x)} - 2ku(x)e^{-2ku(x)} \le 0$$

$$\left(u(x)e^{-2ku(x)}\right)' \le 0$$

If $x_0 < x$, we integrate from x_0 to x,

$$\int_{x_0}^{x}\left(e^{-2ku(x)}u(x)\right)' dx \le 0$$

Multiplying by e^{-2kx} on both sides

$$e^{-2kx}\left(u'(x)-2ku(x)\right)\le 0$$

$$e^{-2kx}u'(x)-2ku(x)e^{-2kx}\le 0$$

$$\left(e^{-2kx}u(x)\right)'\le 0$$

If $x_0<x$, we integrate from x_0 to x

$$\int_{x_0}^{x}\left(e^{-2kx}u(x)\right)'dx\le 0$$

$$\left[e^{-2kx}u(x)\right]_{x_0}^{x}\le 0$$

$$e^{-2kx}u(x)-e^{-2kx_0}u(x_0)\le 0$$

$$e^{-2kx}u(x)\le e^{-2kx_0}u(x_0)$$

$$u(x)\le e^{2kx}\cdot e^{-2kx_0}u(x_0)$$

$$u(x)\le e^{2k(x-x_0)}u(x_0)$$

$$\|\phi(x)\|^2\le e^{2k(x-x_0)}\|\phi(x_0)\|^2$$

Taking square root on both sides.

$$\|\phi(x)\|\le\|\phi(x_0)\|e^{k(x-x_0)} \quad \text{—— (5)}$$

Similarly, taking $-2ku(x)\le u'(x)$, we get

$$\|\phi(x_0)\|e^{-k(x-x_0)}\le\|\phi(x)\| \quad \text{—— (6)}$$

From eqn (5) and (6),

$$\|\phi(x_0)\| e^{-k(x-x_0)} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k(x-x_0)}.$$

where $k = 1 + |a_1| + |a_2|$.

Hence the theorem.

Theorem 1.4: (Uniqueness Theorem)

let α, β be any two constants and let x_0 be any real no. On any interval I containing x_0 there exists atmost one solution ϕ of the initial value

$L(y) = 0, \quad y(x_0) = \alpha, \quad y'(x_0) = \beta.$

Proof:

Suppose ϕ and ψ are two solns. of the initial value problem $L(y) = y'' + a_1 y' + a_2 y = 0$, $y(x_0) = \alpha$, $y'(x_0) = \beta$.

Then we have to prove that $\phi(x) = \psi(x)$ for all x.

let $\chi = \phi - \psi$.

Then $L(\chi) = L(\phi) - L(\psi)$

$= 0$, and $\chi(x_0) = 0$, $\chi'(x_0) = 0$.

Thus $\|\chi(x_0)\| = 0$, and applying the inequalities

$$\|\phi(x_0)\| e^{-k(x-x_0)} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k(x-x_0)}.$$

to χ we see that

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$\|\chi(x)\| = 0$ for all x in I.

This implies $\chi(x) = 0$ for all x in I, or

$$\phi = \psi.$$

Theorem 5:

Let ϕ_1, ϕ_2 be the two solns. of $L(y) = 0$ given by $\phi_1(x) = e^{r_1 x}$, $\phi_2(x) = e^{r_2 x}$ in case $r_1 \neq r_2$ and by $\phi_1(x) = e^{r_1 x}$, $\phi_2(x) = x e^{r_1 x}$ in case $r_1 = r_2$. If c_1, c_2 are any two constants the fun. $\phi = c_1\phi_1 + c_2\phi_2$ is a soln. of $L(y) = 0$ on $-\infty < x < \infty$. Conversely, if ϕ is any soln. of $L(y) = 0$ on $-\infty < x < \infty$, there are unique constants c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$.

Proof:

Let ϕ_1, ϕ_2 be the two solns. of

$$L(y) = 0.$$

To prove : ϕ is a soln. of $L(y) = 0$

Let $\phi = c_1\phi_1 + c_2\phi_2$

$$L(\phi) = L(c_1\phi_1 + c_2\phi_2)$$

$$= L(c_1\phi_1) + L(c_2\phi_2)$$

$$= c_1 L(\phi_1) + c_2 L(\phi_2)$$

$$= c_1(0) + c_2(0)$$

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$L(\phi) = 0.$

$\therefore$ ϕ is a soln. of $L(y) = 0$.

Conversely.

Let ϕ be a soln. of $L(y)=0$. and

x_0 is real, ~~under $f(x_0)=\alpha$, $f'(x_0)=\beta$.~~

let $\phi(x_0) = \alpha$, $\phi'(x_0) = \beta$.

Any soln of initial value problem is of the form $\psi = c_1\phi_1 + c_2\phi_2$.

By uniqueness thm.,

In I there should be atmost one soln.

$\phi = \psi$.

Exercises:

1) Consider the eqn. $y''+y'-6y=0$

a) Compute the soln. ϕ satisfying $\phi(0)=1$, $\phi'(0)=0$

b) Compute the soln ψ satisfying $\psi(0)=0$, $\psi'(0)=1$, c) Compute $\phi(1)$ $\psi(1)$.

soln.

The characteristic polynomial is $r^2+r-6=0$

$(r-2)(r+3)=0$

$r = 2 \quad r = -3$

$\phi(x) = c_1 e^{2x} + c_2 e^{-3x}$

$\phi'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x}$

$\phi(0)=1 \Rightarrow \quad c_1+c_2=1 \quad — ①$

$\phi'(0)=0 \Rightarrow 2c_1-3c_2=0 \quad — ②$

solving ① & ②, $\quad 2c_1+2c_2=2$

$$\begin{array}{r} \cancel{2c_1}+2c_2=2 \\ -\cancel{2c_1}-3c_2=0 \\ \quad\quad + \\ \hline 5c_2=2 \end{array}$$

$$\boxed{c_2=\frac{2}{5}}$$

$c_1+\frac{2}{5}=1 \Rightarrow c_1=1-\frac{2}{5}$

$$\boxed{c_1=\frac{3}{5}}$$

$$\phi(x)=\frac{3}{5}e^{2x}+\frac{2}{5}e^{-3x}$$

H.W (ii) $\psi(x)=c_1e^{2x}+c_2e^{-3x}$

$\psi'(x)=2c_1e^{2x}-3c_2e^{-3x}$

$\psi(0)=0 \Rightarrow \quad c_1e^0+c_2e^0=0$

$c_1+c_2=0 \quad — ①$

$\psi'(0)=1 \Rightarrow \quad 2c_1-3c_2=1 \quad — ②$

① × 2 → $\quad 2c_1+2c_2=0$

$$\begin{array}{r} \cancel{2c_1}+2c_2=0 \\ -\cancel{2c_1}-3c_2=1 \\ + \quad\quad - \\ \hline 5c_2=-1 \end{array}$$

$$\boxed{c_2=\frac{-1}{5}}$$

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$$c_1 - \frac{1}{5} = 0 \Rightarrow \boxed{c_1 = \frac{1}{5}}$$

$$\psi(x) = +\frac{1}{5}e^{2x} + \frac{1}{5}e^{-3x}$$

(iii) $\phi(1) = \frac{3}{5}e^{2(1)} + \frac{2}{5}e^{-3}$

$$\phi(1) = \frac{3}{5}e^{2} + \frac{2}{5}e^{-3}$$

$$\psi(1) = +\frac{1}{5}e^{2} + \frac{1}{5}e^{-3}$$

H.W Find all solns. ϕ of $y'' + y = 0$ satisfying

a) $\phi(0) = 1$, $\phi(\pi/2) = 2$, (b) $\phi(0) = 0$, $\phi(\pi) = 0$

c) $\phi(0) = 0$, $\phi'(\pi/2) = 0$, d) $\phi(0) = 0$, $\phi(\pi/2) = 0$.

Ans:

a) $\phi(x) = \cos x + 2\sin x$

b) $\phi(x) = c\sin x$, c any constant

c) $\phi(x) = c\sin x$, c any constant.

d) $\phi(x) = 0$, all x.

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1. Find the solns of the following initial value Problems:

a) $y'' - 2y' - 3y = 0$, $y(0) = 0$, $y'(0) = 1$

b) $y'' + (4i+1)y' + y = 0$, $y(0) = 0$, $y'(0) = 0$

c) $y'' + (3i-1)y' - 3iy = 0$, $y(0) = 2$, $y'(0) = 0$

d) $y'' + 10y = 0$, $y(0) = \pi$, $y'(0) = \pi^2$

$r^2 + (4i+1)r + 1 = 0$

$r^2 + 4ir + r + 1 = 0$

$r = \frac{-(4i+1) \pm \sqrt{(4i+1)^2 - 4.1.1}}{2}$

$= \frac{-(4i+1) \pm \sqrt{-16+1+8i-4}}{2}$

$\phi(x) = 0 \;\forall x$

$= \frac{-4i-1 \pm \sqrt{-19+8i}}{2}$

$= \frac{-(4i+1) + \sqrt{8i-19}}{2}$, $\frac{-(4i+1) - \sqrt{8i-19}}{2}$

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1.4 Linear dependence and independence

Defn:

* Two fun^{s} ϕ_1, ϕ_2 defined on an interval I are said to be linearly dependent on I if there exist two constants c_1, c_2 not both zero, such that $c_1\phi_1(x) + c_2\phi_2(x) = 0$ for all x in I.
* The fun^{s} ϕ_1, ϕ_2 are said to be linearly independent on I if they are not linearly dependent. Thus ϕ_1, ϕ_2 are linearly independent on I if the only constants c_1, c_2 such that $c_1\phi_1(x) + c_2\phi_2(x) = 0$ for all x in I are the constants $c_1 = 0$, $c_2 = 0$.

The fun^{s} defined by $\phi_1(x) = e^{r_1 x}$, $\phi_2(x) = e^{r_2 x}$ are linearly independent on an interval I. For suppose
$$c_1 e^{r_1 x} + c_2 e^{r_2 x} = 0 \quad \text{--- (1)}$$
for all x in I.

Then, multiplying by $e^{-r_1 x}$, we obtain
$$c_1 + c_2 e^{(r_2 - r_1)x} = 0$$
and differentiating these results
$$c_2(r_2 - r_1)\, e^{(r_2 - r_1)x} = 0$$

Since $r_1 \neq r_2$ and $e^{(r_2-r_1)x}$ is never zero, this implies $C_2=0$. But if $C_2=0$, the relation ① gives $C_1 e^{r_1 x}=0$, or $C_1=0$ also.

$|||^{ly}$, the functions ϕ_1, ϕ_2 defined by $\phi_1(x)=e^{r_1 x}$ and $\phi_2(x)=xe^{r_1 x}$ are linearly independent on any interval I. The proof is same.

If $C_1 e^{r_1 x}+C_2 x e^{r_1 x}=0$ on I,

by multiplying by $e^{-r_1 x}$ we get

$C_1+C_2 x \equiv 0$, and differentiating we obtain

$C_2=0$, and this implies $C_1=0$.

Wronskian Test:

There is a simple test which enables us to tell whether two solns. ϕ_1, ϕ_2 of $L(y)=0$ are linearly independent (or) not.

Let ϕ_1, ϕ_2 be the soln. of $L(y)=0$, the determinant

$$W(\phi_1,\phi_2)=\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

$= \phi_1\phi_2' - \phi_2\phi_1'$ is called the Wronskian of ϕ_1, ϕ_2

Theorem 1.6

Two solns. ϕ_1, ϕ_2 of $L(y)=0$ are linearly independent on an interval I iff $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I.

Proof:

Suppose $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I and let c_1, c_2 be constants such that

$$c_1\phi_1(x) + c_2\phi_2(x) = 0 \quad \text{——} (1)$$

for all x in I. Then also

$$c_1\phi_1'(x) + c_2\phi_2'(x) = 0 \quad \text{——} (2)$$

for all x in I.

For a fixed x the eqns. (1), (2) are linear homogeneous eqns. satisfied by c_1 and c_2.

The determinant of the coeff. is just $W(\phi_1, \phi_2)(x)$ which is non zero.

Hence the matrix representation of the eqn (1) & (2) is

$$\begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the determinant of the coeffs. of c_1, c_2 in eqn. (1) & (2) is just $W(\phi_1, \phi_2)(x)$ which is not zero

$\therefore$ the matrix $\begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix}$ is non-singular.

Hence the above matrix eqn. has unique soln. namely $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

ie) $c_1 = 0, \quad c_2 = 0.$

$\therefore c_1 = 0, c_2 = 0$ is the only soln. of ① & ②.

This proves that ϕ_1, ϕ_2 are linearly independent.

Conversely, assume ϕ_1, ϕ_2 are linearly independent on I. Suppose that there is an x_0 in I such that $W(\phi_1, \phi_2)(x_0) = 0$.

This implies that the system of two eqns:

$$c_1 \phi_1(x_0) + c_2 \phi_2(x_0) = 0 \qquad ③$$

$$c_1 \phi_1'(x_0) + c_2 \phi_2'(x_0) = 0$$

has a soln. c_1, c_2 where atleast one of these nos. is not zero.

Let c_1, c_2 be such a soln. and Consider the fun. $\psi = c_1\phi_1 + c_2\phi_2$. Now $L(\psi) = 0$, and from ③ we see that $\psi(x_0) = 0, \quad \psi'(x_0) = 0$.

From the uniqueness thm. $\psi(x) = 0$ for all x in I,

~~and thus~~ $\Rightarrow c_1\phi_1(x) + c_2\phi_2(x) = 0$ for all x in I.

$\Rightarrow \phi_1$ and ϕ_2 are linearly dependent.

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which is contradiction.

$\therefore W(\phi_1, \phi_2)(x) \neq 0 \quad \forall x$ in I.

Problem:

1. Check the following are linearly dependent (or) independent.

(i) $\phi_1(x) = \sin x, \quad \phi_2(x) = e^{ix}$

soln.

$\phi_1(x) = \sin x, \qquad \phi_2(x) = e^{ix} = \cos x + i\sin x$

$\phi_1'(x) = \cos x, \qquad \phi_2'(x) = -\sin x + i\cos x$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x + i\sin x \\ \cos x & -\sin x + i\cos x \end{vmatrix}$$

$$= -\sin^2 x + i\sin x\cos x - \cos^2 x - i\sin x\cos x$$

$$= -(\sin^2 x + \cos^2 x)$$

$$= -1$$

$W(\phi_1, \phi_2) \neq 0$

$\therefore \phi_1$ and ϕ_2 are linearly independent.

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2) $\phi_1(x) = \cos x$, $\phi_2(x) = 3(e^{ix} + e^{-ix})$

Soln.

$\phi_1(x) = \cos x$ $\qquad$ $\phi_2(x) = 3(e^{ix} + e^{-ix})$

$\phi_1'(x) = -\sin x$ $\qquad$ $= 3(\cos x + i\sin x + \cos x - i\sin x)$

$\phi_2(x) = 6\cos x$

$\phi_2'(x) = -6\sin x$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & 6\cos x \\ -\sin x & -6\sin x \end{vmatrix}$$

$$= -6\sin x\cos x + 6\sin x\cos x$$

$$= 0$$

$\therefore \phi_1$ and ϕ_2 are linearly dependent.

3) $\phi_1(x) = x$, $\qquad \phi_2(x) = |x|$, $\phi_2(x) = x$, if $x \geq 0$.

$\phi_1'(x) = 1$, $\qquad \phi_2'(x) = 1$ $\qquad \phi_2(x) = -x$, if $x < 0$

$$W = \begin{vmatrix} x & +x \\ 1 & 1 \end{vmatrix} \quad W = \begin{vmatrix} x & -x \\ 1 & -1 \end{vmatrix}$$

$W = 0 \quad x - x = 0 \qquad W = -x + x = 0$

$\therefore \phi_1, \phi_2$ are linearly dependent.

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1) $\phi_1(x)=x^2, \quad \phi_2(x)=5x^2$

$\phi_1'(x)=2x \quad \phi_2'(x)=10x.$

$$W=\begin{vmatrix} x^2 & 5x^2 \\ 2x & 10x \end{vmatrix}$$

$$=10x^3-10x^3$$

$$=0.$$

?

$\therefore \phi_1, \phi_2$ are linearly independent.

Theorem: 1.7

Let ϕ_1, ϕ_2 be two solns. of $L(y)=0$ on an interval I and let x_0 be any point I. Then ϕ_1, ϕ_2 are linearly independent on I if and only if $W(\phi_1, \phi_2)(x_0)\neq 0$.

Proof:

If ϕ_1, ϕ_2 are linearly independent on I then

$W(\phi_1, \phi_2)(x)\neq 0$ for all x in I.

By Thm 1.6, In particular $W(\phi_1, \phi_2)(x_0)\neq 0$.

Conversely, Suppose $W(\phi_1, \phi_2)(x_0)\neq 0$ and suppose c_1, c_2 are constants such that

$$c_1\phi_1(x)+c_2\phi_2(x)=0 \quad \forall x \text{ in } I.$$

Then $c_1\phi_1(x_0)+c_2\phi_2(x_0)=0,$

$c_1\phi_1'(x_0)+c_2\phi_2'(x_0)=0,$

Since the determinant of the coeffs. is

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$W(\phi_1, \phi_2)(x_0) \neq 0$, $\therefore$ we obtain $c_1 = c_2 = 0$.
Thus ϕ_1, ϕ_2 are linearly independent on I.

Theorem: 1.8

Let ϕ_1, ϕ_2 be any two linearly independent solns. of $L(y)=0$ on an interval I. Every soln. ϕ of $L(y)=0$ can be written uniquely as $\phi = c_1\phi_1 + c_2\phi_2$ where c_1, c_2 are constants.

Proof;

Let x_0 be a point in I.

Since ϕ_1, ϕ_2 are linearly independent on I, we have ~~$W(\phi_1, \phi_2)(x_0) \neq 0$~~ $W(\phi_1, \phi_2)(x_0) \neq 0$ for some x_0 in I.

Let $\phi(x_0) = \alpha$, $\phi'(x_0) = \beta$, and Consider the two equations

$$c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha$$

$$c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta$$

Since the determinant of the coefficients of c_1, c_2 is just $W(\phi_1, \phi_2)(x_0) \neq 0$, there is a unique pair of constants c_1, c_2 satisfying those equations.

choose c_1, c_2 to be these constants.
Then the fun: $\psi = c_1\phi_1 + c_2\phi_2$ is such that
$\psi(x_0) = \phi(x_0)$, $\psi'(x_0) = \phi'(x_0)$ and $L(\psi) = 0$.
From the uniqueness theorem, it follows that
$\psi = \phi$ on I.
ie) $\phi = c_1\phi_1 + c_2\phi_2$, where c_1 and c_2 are constants.

Example:

Consider the eqn: $y'' + y = 0$.
Its characteristic poly. is $r^2+1=0$

$$r^2 = -1$$
$$r = \pm i$$

$\phi_1(x) = e^{ix}$, $\phi_2(x) = e^{-ix}$ which are linearly independent,
~~which are linearly independent.~~
Since $\phi_1'(x) = ie^{ix}$ $\phi_2'(x) = -ie^{-ix}$

$$W = \begin{vmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{vmatrix}$$
$$= -ie^{0} - ie^{ix-ix}$$
$$= -i - i$$
$$= -2i$$

$W \neq 0$, for x.

But it also has the two linearly independent solns of $\cos x, \sin x$.

5) A formula for the Wronskian:

There is a convenient formula for the Wronskian of two solns. of $L(y)=0$, which results from the fact that $W(\phi_1, \phi_2)$ satisfies a first order linear equation.

Theorem 1.9:

If ϕ_1, ϕ_2 are two solns. of $L(y)=0$ on an interval I containing a point x_0, then
$W(\phi_1, \phi_2)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2)(x_0)$.

Proof:

Let ϕ_1, ϕ_2 be two solns. of $L(y)=0$. Then we have

$$\phi_1'' + a_1\phi_1' + a_2\phi_1 = 0, \quad \text{——} (1)$$

$$\phi_2'' + a_1\phi_2' + a_2\phi_2 = 0, \quad \text{——} (2)$$

Multiplying eqn (1) by $-\phi_2$ and eqn (2) by ϕ_1, and adding we obtain

(1) $\times -\phi_2 \Rightarrow$ $\quad -\phi_2\phi_1'' + a_1\phi_1'(-\phi_2) - a_2\phi_1\phi_2 = 0$

(2) $\times \phi_1$ $\quad \phi_1\phi_2'' + a_1\phi_2'\phi_1 + a_2\phi_1\phi_2 = 0$

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$(\phi_1\phi_2'' - \phi_2\phi_1'') + (\phi_2'\phi_1 - \phi_1'\phi_2)a_1 = 0$

We notice that if $W = W(\phi_1, \phi_2)$,

$W = \phi_1\phi_2' - \phi_2\phi_1'$ and $W' = \phi_1\phi_2'' + \cancel{\phi_1'\phi_2'} - \cancel{\phi_2'\phi_1'} - \phi_2\phi_1''$

$W' = \phi_1\phi_2'' - \phi_1''\phi_2$

Thus W satisfies the first order eqn

$$W' + a_1 W = 0$$

A.E is $r + a_1 = 0 \Rightarrow r = -a_1$

Hence the soln. is $W(x) = c\,e^{-a_1 x}$, ——— ③

where c is some constant.

Setting $x = x_0$ we see that $W(x_0) = c\,e^{-a_1 x_0}$,

(or) $c = W(x_0)\,e^{a_1 x_0}$,

sub. c value in eqn ③

$$W(x) = W(x_0)\,e^{a_1 x_0} \cdot e^{-a_1 x}$$

$$W(x) = e^{-a_1(x - x_0)}\,W(x_0)$$

Hence prove the theorem.

1.6 Non-homogeneous eqn. of order two

When solving the non-homogeneous linear differential eqn. $L(y) = y'' + a_1 y' + a_2 y = b(x)$, where b is some continuous fun.

e^{ax} - Replace D by a $\qquad \frac{1}{(D-a)^r} = \frac{x^r e^{ax}}{r!}$

$\sin ax, \cos ax$ - D^2 by $-a$

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6. The non homogeneous eqn. of order two

The second General ODE non homogeneous is given by $L(y) = y'' + a_1 y' + a_2 y = b(x)$.

1) Solve $y'' - y' - 2y = e^{-x}$

Soln. Let $L(y) = y'' - y' - 2y = 0$

The characteristic eqn is $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0$$

$$r = 2, -1$$

$$\phi_1 = e^{2x}, \quad \phi_2 = e^{-x}$$

The soln. is $\phi = c_1\phi_1 + c_2\phi_2$

$$= c_1 e^{2x} + c_2 e^{-x}$$

$$P.I = \frac{1}{D^2 - D - 2} e^{-x}$$

$$= \frac{1}{(-1)^2 - (-1) - 2} e^{-x}$$

$$= \frac{1}{1+1-2} e^{-x}$$

$$= \frac{1}{(D-2)(D+1)} e^{-x}$$

$$= \frac{x e^{-x}}{D-2} = \frac{x e^{-x}}{-3}$$

$\frac{x\, f(x)}{f'(D)}$

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$\phi = C.F + P.I = c_1 e^{2x} + c_2 e^{-x} - \frac{x}{3} e^{-x}.$

2) Solve $y'' + 4y = \cos x.$

Soln. $L(y) = y'' + 4y = 0$

The characteristic eqn. is ~~$D^2 + 4y = 0$~~ $r^2 + 4 = 0$

$r^2 = -4 \Rightarrow r = \pm 2i$

$C.F = C_1 \cos 2x + C_2 \sin 2x.$

$P.I = \frac{1}{D^2+4} \cos x$

$= \frac{1}{-1+4} \cos x$

$= \frac{1}{3} \cos x$

The soln. is $\phi = c_1\phi_1 + c_2\phi_2 + P.I$

$= C_1 \cos 2x + C_2 \sin 2x.$

3) Solve $y'' + 9y = \sin 3x$

Soln

Let $L(y) = y'' + 9y = 0$

The characteristic eqn. is $r^2 + 9 = 0$

$r^2 = -9 \Rightarrow r = \pm 3i$

$C.F = C_1 \cos 3x + C_2 \sin 3x$

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$$P.I = \frac{1}{D^2+9}\sin 3x$$

$$= \frac{1}{-9+9}\sin 3x$$

$$= \frac{1}{0}\sin 3x \quad \text{failure}$$

$$= \frac{x \sin 3x}{f'(D)}$$

$$= \frac{x \sin 3x}{2D}$$

$$= \frac{x\, D \sin 3x}{2D^2}$$

$$\cancel{= \frac{x \cos 3x}{2\cdot(-9)}}$$

$$= \frac{-x \cos 3x}{\cancel{2}}$$

$$= \frac{x(\cancel{3}\cos 3x)}{2(-\cancel{9}_{3})}$$

$$P.I = \frac{x \cos 3x}{-6}$$

General soln is

$$y = C_1 \cos 3x + C_2 \sin 3x - \frac{x \cos 3x}{6}$$

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(or)

$$P.I = \frac{\sin 3x}{D^2+9} = \text{Imaginary Part of } \frac{e^{i3x}}{(D+3i)(D-3i)}$$

$$= \text{Im. Part of } \frac{e^{i3x}}{6i(D-3i)}$$

$$= \text{Im. part of } \frac{x e^{i3x}}{6i}$$

$$= \text{Im Part of } \frac{x}{6i}(\cos 3x + i\sin 3x)$$

$$= \text{" " } \frac{x}{6}(-i\cos 3x + \sin 3x)$$

$$= \frac{-x\cos 3x}{6}.$$

4) Solve $y'' - 7y' + 6y = \sin x$.

Soln. Let $L(y) = y'' - 7y' + 6y = 0$

The characteristic eqn. is, $r^2 - 7r + 6 = 0$

$(r-1)(r-6) = 0 \Rightarrow r = 1, 6$

$C.F = C_1 e^{x} + C_2 e^{6x}$

$$P.I = \frac{1}{D^2 - 7D + 6}\sin x$$

$$= \frac{1}{-1 - 7D + 6}\sin x$$

$$= \frac{1}{5 - 7D}\sin x$$

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$$= \frac{1}{5-7D} \cdot \frac{5+7D}{5+7D} \sin x$$

$$= \frac{5\sin x + 7\cos x}{25-49D^2}$$

$$= \frac{5\sin x + 7\cos x}{25-49(-1)}$$

$$= \frac{5\sin x + 7\cos x}{25+49}$$

$$= \frac{5\sin x + 7\cos x}{74}$$

General soln is $\phi = c_1 e^{x} + c_2 e^{6x} + \frac{5\sin x + 7\cos x}{74}$

Types

$(D^2 = -a^2)$

case (i) $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$ if $f(-a^2) \neq 0$

$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$ if $f(-a^2) \neq 0$

case (ii) $f(-a^2) = 0$

$$\frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax$$

$$\frac{1}{D^2+a^2} \cos ax = \frac{x\sin ax}{2a}$$

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Type: e^{ax}.

If $f(a) \neq 0$, $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ $(D=a$

If $f(a) = 0$, $f(D)$ must possess a factor of $(D-a)^r$.

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax} \qquad r = 1, 2, \ldots$$

If one factor is zero

$$\frac{1}{f(D)} e^{ax} = \frac{x e^{ax}}{(a-m_2)} \qquad (a = m_1)$$

If two factor is zero

$$\frac{1}{f(D)} e^{ax} = \frac{x^2}{2!} e^{ax}$$

Theorem: 1.10

Let $b(x)$ be continuous on I. Every soln. ϕ of $L(y) = b(x)$ on I can be written as $\psi = \psi_p + c_1\phi_1 + c_2\phi_2$ where ψ_p is a particular soln., ϕ_1, ϕ_2 are linearly independent soln. of $L(y) = 0$ and c_1, c_2 are constants.

The particular soln

$$\psi_p = \int_{x_0}^{x} \frac{\phi_1(t)\phi_2(x) - \phi_1(x)\cdot\phi_2(t)}{W(\phi_1, \phi_2)(t)} \, b(t) \, dt.$$

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Conversely, such ψ is a soln of $L(y)=b(x)$

Proof:

The second order eqn of non homogeneous eqn is $L(y)=y''+a_1y'+a_2y=b(x)$
where b is som cts fun on an interval I
Suppose we know that ψ_p is a Particular soln of this eqn and that ψ is any other soln.
Then $L(\psi-\psi_p)=L(\psi)-L(\psi_p)=b-b=0$ on I
$\Rightarrow$ $\psi-\psi_p$ is a soln of the homogeneous eqn $L(y)=0$.
$\therefore$ if ϕ_1, ϕ_2 are linearly independent solns of $L(y)=0$, there are unique constants c_1, c_2 such that $\psi-\psi_p=c_1\phi_1+c_2\phi_2$
Every soln ψ of $L(y)=b(x)$ can be written as
$$\psi=\psi_p+c_1\phi_1+c_2\phi_2$$
and the problem of finding all solns of $L(y)=b(x)$ reduces to finding a particular one ψ_p, and two linearly independent solns ϕ_1, ϕ_2 of $L(y)=0$.
If $L(\psi_p)=b$, and $L(\phi_1)=L(\phi_2)=0$, and c_1, c_2 are any constants, then
$$\psi=\psi_p+c_1\phi_1+c_2\phi_2$$

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satisfies $L(\psi) = b$.

Every soln. of $L(y)=0$ is of the form $c_1\phi_1 + c_2\phi_2$ where c_1, c_2 are constants, and ϕ_1, ϕ_2 are linearly independent solns.

Such a fun. $c_1\phi_1 + c_2\phi_2$ can not be a soln. of $L(y)=b(x)$ unless $b(x)=0$ on I. However, suppose we allow c_1, c_2 to become funs. u_1, u_2 on I, then there is a soln. of $L(y)=b(x)$ of the form $u_1\phi_1 + u_2\phi_2$.

Proof of theorem:

Suppose we have a soln. of $L(y)=b(x)$ is of the form $u_1\phi_1 + u_2\phi_2$ where u_1 and u_2 are the funs. of x and ϕ_1, ϕ_2 are the solns. of $L(y)=0$.

Let $L(y) = b(x)$

$L(y) = y'' + a_1 y' + a_2 y = b(x)$

Since $u_1\phi_1 + u_2\phi_2$ is the soln. of $L(y)=b(x)$, then it satisfies the eqn.

$$(u_1\phi_1 + u_2\phi_2)'' + a_1(u_1\phi_1 + u_2\phi_2)' + a_2(u_1\phi_1 + u_2\phi_2) = b(x) \quad (*)$$

$$D(u_1\phi_1 + u_2\phi_2) = D(u_1\phi_1) + D(u_2\phi_2)$$
$$= u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2 \quad \text{--- (1)}$$

$$D^2(u_1\phi_1 + u_2\phi_2) = D(u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2)$$
$$= u_1\phi_1'' + u_1'\phi_1' + u_1''\phi_1 + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + u_2''\phi_2 + u_2'\phi_2' \quad (2)$$

$$(*) \Rightarrow b(x) = u_1\phi_1'' + u_1'\phi_1' + u_1''\phi_1 + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + u_2''\phi_2 + u_2'\phi_2' + a_1u_1\phi_1' + a_1u_1'\phi_1 + a_1u_2\phi_2' + a_1u_2'\phi_2 + a_2u_1\phi_1 + a_2u_2\phi_2$$

$$b(x) = u_1(\phi_1'' + a_1\phi_1' + a_2\phi_1) + u_2(\phi_2'' + a_1\phi_2' + a_2\phi_2) + a_1(u_1'\phi_1 + u_2'\phi_2) + 2(u_1'\phi_1' + u_2'\phi_2') + \phi_1u_1'' + \phi_2u_2''$$

$$b(x) = u_1 L(\phi_1) + u_2 L(\phi_2) + a_1(u_1'\phi_1 + \phi_2u_2') + 2(u_1'\phi_1' + u_2'\phi_2') + \phi_1u_1'' + \phi_2u_2''$$

Since $L(\phi_1)$ and $L(\phi_2)$ are both zero, we get

$$a_1(u_1'\phi_1 + u_2'\phi_2) + 2(u_1'\phi_1' + u_2'\phi_2') + \phi_1u_1'' + \phi_2u_2'' = b(x) \quad (3)$$

if $\phi_1u_1' + \phi_2u_2' = 0$, then — (4)

$$[\phi_1u_1' + \phi_2u_2']' = 0$$

$$\phi_1'u_1' + \phi_1u_1'' + \phi_2u_2'' + \phi_2'u_2' = 0 \quad \text{--- (5)}$$

$$-[\phi_1'u_1' + \phi_2'u_2'] = +[\phi_2u_2'' + \phi_1u_1'']$$

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③ $\Rightarrow a_1(0) + 2(u_1'\phi_1' + u_2'\phi_2') - \phi_1' u_1' + \phi_2' u_2' = b(x)$

$u_1'\phi_1' + u_2'\phi_2' = b(x)$ ——— ⑥

Equation ④ & ⑥ denotes the two linear eqn. for u_1', u_2'.

For unique soln. solving ④ & ⑥

From eqn. ④ $u_2' = \dfrac{-\phi_1 u_1'}{\phi_2}$,

$$u_1' = \frac{-\phi_2 u_2'}{\phi_1}$$

sub. u_2' in eqn ⑥

Eqn ⑥ $\Rightarrow u_1'\phi_1' - \dfrac{\phi_1 u_1'}{\phi_2}\phi_2' = b(x)$

$$u_1'\phi_1'\phi_2 - \phi_1 u_1'\phi_2' = b\phi_2$$

$$u_1'(\phi_1'\phi_2 - \phi_1\phi_2') = b\phi_2$$

$$-u_1'[\phi_1\phi_2' - \phi_1'\phi_2] = b\phi_2$$

$$-u_1' W(\phi_1, \phi_2) = b\phi_2$$

$$u_1' = \frac{-b\phi_2}{W(\phi_1, \phi_2)}$$

sub. u_1' in eqn ⑥

Eqn. ⑥ $\Rightarrow \dfrac{-\phi_2 u_2'}{\phi_1}\phi_1' + u_2'\phi_2' = b(x)$

$$-\phi_2 u_2'\phi_1' + u_2'\phi_1\phi_2' = b(x)\phi_1$$

$$u_2'[-\phi_2\phi_1' + \phi_1\phi_2'] = b(x)\phi_1$$

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$$u_2' \cdot W[\phi_1, \phi_2] = b\phi_1$$

$$u_2' = \frac{b\phi_1}{W[\phi_1, \phi_2]}$$

Integrating the above eqn. w.r.to t, we get

$$u_1(x) = \int_{x_0}^{x} \frac{-\phi_2(t)\, b(t)}{W(\phi_1, \phi_2)\, t}\, dt$$

$$u_2(x) = \int_{x_0}^{x} \frac{\phi_1(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt$$

Thus the soln. $\psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$

$$\psi_p = \left(\int_{x_0}^{x} \frac{-\phi_2(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt \right) \phi_1(x) + \int_{x_0}^{x} \frac{\phi_1(t) b(t)}{W(\phi_1, \phi_2)} \phi_2(x).$$

$$\psi_p = \int_{x_0}^{x} \left[\frac{\phi_1(t)\, \phi_2(x) - \phi_2(t)\phi_1(x)}{W(\phi_1, \phi_2)(t)} \right] b(t)\, dt.$$

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Theorem: ~~1.1~~

1) solve $y'' - y' - 2y = e^{-x}$.

→ The particular soln ψ_p of the non-homogeneous eqn is of the form

$$\psi_p(x) = u_1(x) e^{+2x} + u_2(x) e^{-x}.$$

→ The characteristic polynomial is $r^2 - r - 2 = 0$.

$$(r-2)(r+1) = 0.$$

$$r = 2, \quad r = -1 \qquad \Rightarrow \phi_1(x) = e^{2x}, \ \phi_2(x) = e^{-x}$$

→ To find u_1' and u_2'. $\qquad \phi_1'(x) = 2e^{2x}, \ \phi_2'(x) = -e^{-x}$

u_1' and u_2' are satisfies the eqn.

$$\phi_1 u_1' + \phi_2 u_2' = 0 \text{ and } \phi_1' u_1' + \phi_2' u_2' = b.$$

$$e^{2x} u_1' + e^{-x} u_2' = 0 \qquad \text{---} \ (*)$$

$$2e^{2x} u_1' - e^{-x} u_2' = e^{-x}$$

~~Solving the above eqn. we get~~

$$3e^{2x} u_1' = e^{-x}$$

$$u_1' = \frac{e^{-x} \cdot e^{-2x}}{3}$$

$$u_1' = \frac{e^{-3x}}{3}$$

$$(*) \Rightarrow e^{2x} \frac{e^{-3x}}{3} + e^{-x} u_2' = 0$$

$$\frac{e^{-x}}{3} + e^{-x} u_2' = 0$$

$$e^{-x} u_2' = -\frac{e^{-x}}{3}$$

$$u_2' = -\frac{1}{3}$$

$$u_1 = \int u_1' \, dx = \int \frac{e^{-3x}}{3} dx = \frac{-e^{-3x}}{9}$$

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$$u_1(x) = \frac{-e^{-3x}}{9}$$

and $u_2 = \int u_2' \, dx = \int -\frac{1}{3} \, dx = -\frac{x}{3}$

$\therefore \Rightarrow \quad y_p(x) = \frac{-e^{-3x}}{9} e^{+2x} - \frac{x}{3} e^{-x}$

$$= \frac{-e^{-x}}{9} - \frac{x}{3} e^{-x}.$$

General soln is

$$y = y_p + c_1 e^{2x} + c_2 e^{-x}$$

$$= c_1 e^{2x} + c_2 e^{-x} - \frac{e^{-x}}{9} - \frac{x}{3} e^{-x}$$