MATHEMATICAL CONCEPTS FOR ELECTRONICS

INTRODUCTION

NUMERICAL ANALYSIS

Numerical Analysis is the branch of mathematics that provides tools and methods for solving mathematical problems in numerical form.

In numerical analysis we are mainly interested in implementation and analysis of numerical algorithms for finding an approximate solution to a mathematical problem.

NUMERICAL ALGORITHM

A complete set of procedures which gives an approximate solution to a mathematical problem.

CRITERIA FOR A GOOD METHOD

1) **Number of computations i.e. Addition, Subtraction, Multiplication and Division.**
2) **Applicable to a class of problems.**
3) **Speed of convergence.**
4) **Error management.**
5) **Stability.**

STABLE ALGORITHM

Algorithm for which the cumulative effect of errors is limited, so that a useful result is generated is called stable algorithm. Otherwise Unstable.

NUMERICAL STABILITY

Numerical stability is about how a numerical scheme propogate error.

NUMERICAL ITERATION METHOD

A mathematical procedure that generates a sequence of improving approximate solution for a class of problems i.e. the process of finding successive approximations.

ALGORITHM OF ITERATION METHOD

A specific way of implementation of an iteration method, including to termination criteria is called algorithm of an iteration method.

In the problem of finding the solution of an equation, an iteration method uses as initial guess to generate successive approximation to the solution.

CONVERGENCE CRITERIA FOR A NUMERICAL COMPUTATION

If the method leads to the value close to the exact solution, then we say that the method is convergent otherwise the method is divergent.i.e. $\lim_{n\to\infty} x_n = r$

ROUNDING

For $x \epsilon R$**; f(x) is an element of "F" nearest to "x" and the transformation x→f(x) is called Rounding (to nearest).**

Why we use numerical iterative methods for solving equations?

As analytic solutions are often either too tiresome or simply do not exist, we need to find an approximate method of solution. This is where numerical analysis comes into picture.

LOCAL CONVERGENCE

An iterative method is called locally convergent to a root, if the method converges to root for initial guesses sufficiently close to root.

RATE OF CONVERGENCE OF AN ITERATIVE METHOD

Suppose that the sequence (x_k) converges to "r" then the sequence (x_k) is said to converge to "r" with order of convergence "a" if there exist a positive constant "p" such that

$$\lim_{k\to\infty} \frac{|x_{k+1}-r|}{|x_k-r|^a} = \lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k^{\alpha}} = p$$

Thus if $a = 1$**, the convergence is linear. If** $a = 2$**, the convergence is quadratic and so on.Where the number "*a*" is called convergence factor.**

REMARK

- **Rate of convergence for fixed point iteration method is linear.**
- **Rate of convergence for Newton Raphson method is quadratic.**
- **Rate of convergence for Secant method is Super linear.**

ORDER OF CONVERGENCE OF THE SEQUENCE

Let (x_0, x_1,x_2,.......)be a sequence that converges to a number "a" and set $\mathcal{E}_n$=a - x_n

If there exist a number "k" and a positive constant "c" such that $\lim_{n\to\infty} \frac{|\epsilon_n+1|}{|\epsilon_n|^k} = c$

Then "k" is called order of convergence of the sequence and "c" the asymptotic error constant.

CONSISTENT METHOD

Let $x\epsilon[a,b]$, $y\epsilon R^d$ and the function $f:[a,b]\times R^d\times R_+\to R^d$ may be thought of as the approximate increment per unit step, Or the approximate difference quotient and it defines the method and consider T(x,y:h) is truncation error then the method "f" is called consistent if T(x,y:h)→0 as h→0 uniformly for (x,y) ϵ $[a,b]\times R^d$

PRECISION

Precision mean how close are the measurements obtained from successive iterations.

ACCURACY

Accuracy means how close are our approximations from exact value.

DEGREE OF ACCURACY OF A QUADRATURE FORMULA

It is the largest positive integer "n" such that the formula is exact for "x^k " for each (k=0,1,2,……n). i.e. Polynomial integrated exactly by method.

CONDITION OF A NUMERICAL PROBLEM

A problem is well conditioned if small change in the input information causes small change in the output. Otherwise it is ill conditioned.

STEP SIZE, STEP COUNT, INTERVAL GAP

The common difference between the points i.e. h = $\frac{b-a}{n}$ = t_{i+1} - t_i is called $step - size$.

ERROR ANALYSIS

ERROR

Error is a term used to denote the amount by which an approximation fails to equal the exact solution. $Error = Exact\ solution - Approximation$

SOURCE OF ERRORS

Numerically computed solutions are subject to certain errors. Mainly there are three types of errors

1. **Inherent errors**
2. **Truncation errors**
3. **Round Off errors**

INHERENT (EXPERIMENTAL) ERRORS

Errors arise due to assumptions made in the mathematical modeling of problems. Also arise when the data is obtained from certain physical measurements of the parameters of the problem i.e. errors arising from measurements.

TRUNCATION ERRORS

Errors arise when approximations are used to estimate some quantity.

These errors corresponding to the facts that a finite (infinite) sequence of computational steps necessary to produce an exact result is "truncated" prematurely after a certain number of steps.

<u>How Truncation error can be removed?</u>

Use exact solution.

Error can be reduced by applying the same approximation to a larger number of smaller intervals or by switching to a better approximation.

ROUND OFF ERRORS

Errors arising from the process of rounding off during computations.

These are also called "$chopping$" **i.e. discarding all decimals from some decimals on.**

RELATIVE ERRORS

If "ā" is an approximate value of a quantity whose exact value is "a" then relative error (ϵ_r) of "ā" is defined by $|\epsilon_r| = \frac{|error|}{|true\ value|} = \frac{|\epsilon|}{|a|}$

EXAMPLE

Consider $\sqrt{2}$**=1.414213……. upto four decimal places then** $\sqrt{2}$**=1.4142+ errors**

|error|=|1.4142-1.41421|=0.00001 taking 1.4142 as true or exact value.

Hence $\epsilon_r = \frac{0.00001}{1.4142}$

REMARK

I. $\epsilon_r \approx \frac{\epsilon}{\bar{a}}$ **if |** ϵ **| is much less than |ā|**
II. **We may also introduce the quantity "** γ**= a-ā = -** ϵ**" and called it the "correction"**
III. $True\ value\ =\ Approximate\ value\ +\ Correction$

ABSOLUTE ERROR

If "ā" is an approximate value of a quantity whose exact value is "a" then the difference " ϵ **=ā-a" is called absolute error of "a".**

- **ā =a +** ϵ

EXAMPLE

If ā = 10.52 is an approximation to a = 10.5 then the error is ϵ **= 0.02**

ERROR BOUND

It is a number "β" for "ā" such that |ā-a|≤β i.e. | ϵ **|≤β**

PROBABLE ERROR

This is an error estimate such that the actual error will exceed the estimate with probability one – half.

In other words, the actual error is as likely to be greater than the estimate as less. Since this depends upon the error distribution, it is not an easy target and a rough substitute is often used $\sqrt{n}\,\epsilon$ **with "**ϵ**" the maximum possible error.**

INPUT ERROR

Error arises when the given values ($y_0 = f(x_0)$, y_1, y_2,..........y_n) are inexact as experimental or computed values usually are.

LOCAL ERROR

This is the error after first step.

$\epsilon_{i+1} = x(t_0+h) - x_1$

The Local Error is the error introduced during one operation of the iterative process.

GLOBAL ERROR

This is the error at n-step.

$\epsilon_n = x(t_n) - x_n$

The Global Error is the accumulation error over many iterations.

Note that the Global Error is not simply the sum of the Local Errors due to the non-linear nature of many problems although often it is assumed to be so, because of the difficulties in measuring the global error.

LOCAL TRUNCATION ERROR

It is the ratio of local error by step size.

$$\text{LTE} = \frac{LOCAL\ ERROR}{STEP\ SIZE}$$

REMARK : Floating point numbers are not equally spaced.

SOLUTION OF NON-LINEAR EQUATIONS

ROOTS (SOLUTION) OF AN EQUATION OR ZEROES OF A FUNCTION

Those values of "x" for which f(x) = 0 is satisfied are called root of an equation. Thus "a" is root of f(x) = 0 iff f(a) = 0

DEFLATION: **It is a technique to compute the other roots of f(x) = 0**

ZERO OF MULTIPLICITY

A solution "p" of f(x) = 0 is a zero of multiplicity "m" of "f" if for "x ≠ p" we can write
$f(x) = (x-p)^m q(x)$ where "$\lim_{k \to p} q(x) \neq 0$"

ALGEBRAIC EQUATION

The equation f(X) = 0 is called an algebraic equation if it is purely a polynomial in "x".
e.g. $x^3+5x^2-6x+3 = 0$

TRANSCENDENTAL EQUATION

The equation f(x) = 0 is called transcendental equation if it contains Trigonometric, Inverse trigonometric, Exponential, Hyperbolic or Logarithmic functions.e.g.

i. **M = e-esinx**

ii. **ax^2+ log(x-3) +exsinx = 0**

PROPERTIES OF ALGEBRAIC EQUATIONS

1. **Every algebraic equation of degree "n" has "n" and only "n" roots.e.g.**

   $x^2 - 1=0$ has distinct roots i.e. 1, -1

   $x^2+2x+1 = 0$ has repeated roots i.e. -1, -1

   $x^2+1 = 0$ has complex roots i.e. +i, -i

2. **Complex roots occur in pair. i.e. (a+bi) and (a-bi) are roots of f(x)=0**
3. **If x = a is a root f(x)=0, a polynomial of degree "n" then (x-a) is factor of f(x)=0 on dividing f(x) by (x-a) we obtain polynomial of degree (n-1).**

DISCARTS RULES OF SIGNS

The number of positive roots of an algebraic equation f(x)=0 with the real coefficient cannot exceed the number of changes in sign of the coefficient in f(x)=0.

e.g. $x^3-3x^2+4x-5=0$ changes its sign 3-time, so it has 3 roots.

Similarly, the number of negative roots of f(x)=0 cannot exceed the number of changes in sign of the coefficient of f(-x) =0

e.g. $-x^3-3x^2-4x-5=0$ does not changes its sign, so it has no negative roots.

REMARK

There are two types of methods to find the roots of Algebraic and Transcendental equations.
(i) DIRECT METHODS (ii) INDIRECT (ITERATIVE) METHODS

DIRECT METHODS

1. **Direct methods give the exact value of the roots in a finite number of steps.**
2. **These methods determine all the roots at the same time assuming no round off errors.**
3. **In the category of direct methods; Elimination Methods are advantageous because they can be applied when the system is large.**

INDIRECT (ITERATIVE) METHODS

1. **These are based on the concept of successive approximations. The general procedure is to start with one or more approximation to the root and obtain a sequence of iterates "x" which in the limit converges to the actual or true solution to the root.**
2. **Indirect Methods determine one or two roots at a time.**
3. **Rounding error have less effect**
4. **These are self-correcting methods.**
5. **Easier to program and can be implemented on the computer.**

REMEMBER: Indirect Methods are further divided into two categories

I. BRACKETING METHODS II. OPEN METHODS

BRACKETING METHODS

These methods require the limits between which the root lies. e.g. Bisection method, False position method.

OPEN METHODS

These methods require the initial estimation of the solution. e.g. Newton Raphson method.

ADVANTAGES AND DISADVANTAGES OF BRACKETING METHODS

Bracket methods always converge.

The main disadvantage is, if it is not possible to bracket the root, the method cannot applicable.

GEOMETRICAL ILLUSTRATION OF BRACKET FUNCTIONS

In these methods we choose two points "x_n" and "x_{n-1}" such that $f(x_n)$ and $f(x_{n-1})$ are of opposite signs.

Intermediate value property suggests that the graph of "y=f(x)" crosses the x-axis between these two points, therefore a root (say) "$x=x_r$" lies between these two points.

REMARK

Always set your calculator at radian mod while solving Transcendental or Trigonometric equations.

<u>How to get first approximation?</u>

We can find the approximate value of the root of f(x)=0 by "Graphical method" or by "Analytical method".

INTERMEDIATE VALUE THEOREM

Suppose "f" is continuous on [a, b] and $f(a) \neq f(b)$ then given a number "ƛ" that lies between f(a) and f(b) then there exist a point "c" such that a<c<b with $f(c) = ƛ$

BISECTION METHOD

Bisection method is one of the bracketing methods. It is based on the "Intermediate value theorem"

The idea behind the method is that if f(x) ∈ C [a, b] and f(a).f(b)<0 then there exist a root "c ∈ (a,b)" such that "f(c)=0"

This method also known as BOLZANO METHOD (or) BINARY SECTON METHOD.

ALGORITHM

For a given continuous function f(x)

1. **Find a ,b such that f(a).f(b)<0 (this means there is a root "r ∈ (a,b)" such that f(r)=0**
2. **Let c = $\frac{a+b}{2}$ (mid-point)**
3. **If f(c)=0; done (lucky!)**
4. **Else; check if $f(c).f(a) < 0$ or $f(c).f(b) < 0$**
5. **Pick that interval [a, c] or [c, b] and repeat the procedure until stop criteria satisfied.**

STOP CRITERIA

1. **Interval small enough.**
2. **$|f(c_n)|$ almost zero**
3. **Maximum number of iteration reached**
4. **Any combination of previous ones**

CONVERGENCE CRITERIA

No. of iterations needed in the bisection method to achieve certain accuracy

Consider the interval $[a_0, b_0]$,, $c_0 = \frac{a_0+b_0}{2}$ and let $r \in (a_0, b_0)$ be a root then the error is $\epsilon_0 = |r-c_0| \leq \frac{b_0-a_0}{2}$

Denote the further intervals as $[a_n, b_n]$ for iteration number "n" then

$$\epsilon_n = |r-c_n| \leq \frac{b_n-a_n}{2} \leq \frac{b_0-a_0}{2^{n+1}} = \frac{\epsilon_0}{2^n}$$

If the error tolerance is "ϵ" we require "$\epsilon_n \leq \epsilon$" then $\frac{b_0-a_0}{2^{n+1}} \leq \epsilon$

After taking logarithm $\Longrightarrow$ $\log(b_0-a_0) - n\log 2 \leq \log(2\epsilon)$

$$\Longrightarrow \frac{\log(b_0-a_0) - \log(2\epsilon)}{\log 2} \leq n \quad \Longrightarrow \frac{\log(b-a) - \log 2\epsilon}{\log 2} \leq n$$ **(which is required)**

MERITS OF BISECTION METHOD

1. **The iteration using bisection method always produces a root, since the method brackets the root between two values.**
2. **As iterations are conducted, the length of the interval gets halved. So one can guarantee the convergence in case of the solution of the equation.**
3. **Bisection method is simple to program in a computer.**

DEMERITS OF BISECTION METHOD

1. **The convergence of bisection method is slow as it is simply based on halving the interval.**
2. **Cannot be applied over an interval where there is discontinuity.**
3. **Cannot be applied over an interval where the function takes always value of the same sign.**
4. **Method fails to determine complex roots (give only real roots)**
5. **If one of the initial guesses "a_0" or "b_0" is closer to the exact solution, it will take larger number of iterations to reach the root.**

EXAMPLE

Solve x^3-9x+1 for roots between x=2 and x=4

SOLUTION

X	2	4
f(x)	-9	29

Since f (2). f (4) <0 therefore root lies between 2 and 4

(1) $x_r = \frac{2+4}{2} = 3$ so f(3) = 1 (+ve)

(2) For interval [2,3] ; $x_r = \frac{2+3}{2} = 2.5$
f (2.5) = -5.875 (-ve)

(3) For interval [2.5,3]; x_r = (2.5+3)/2 = 2.75
f (2.75) = -2.9534 (-ve)

(4) For interval [2.75,3]; x_r = (2.75+3)/2 = 2.875
f (2.875) = -1.1113 (-ve)

(5) For interval [2.875,3]; x_r = (2.875+3)/2 = 2.9375
f (2.9375) = -0.0901 (-ve)

(6) For interval [2.9375,3]; x_r = (2.9375+3)/2 = 2.9688
f (2.9688) = +0.4471 (+ve)

(7) For interval [2.9375,2.9688]; x_r = (2.9375+2.9688)/2 = 2.9532
f (2.9532) = +0.1772 (+ve)

(8) For interval [2.9375,2.9532]; x_r = (2.9375+2.9532)/2 = 2.9453
f (2.9453) = 0.1772
Hence root is 2. 9453 because roots are repeated.

EXAMPLE

Use bisection method to find out the roots of the function describing to drag coefficient of parachutist given by

$f(c) = \frac{667.38}{c}$ [1-exp(-0.146843c)]-40 Where "c=12" to "c=16" perform at least two iterations.

SOLUTION

Given that $f(c) = \frac{667.38}{c}$ [1-exp(-0.146843c)]-40

X	12	13	14	15
f(x)	6.670	3.7286	1.5687	-0.4261

Since f (14). f (15) <0 therefore root lie between 14 and 15

$X_r = \frac{14+15}{2} = 14.5$ So f(14.5) = 0.5537

Again f (14.5). f (15) <0 therefore root lie between 14.5 and 15

$x_r = \frac{14.5+15}{2} = 14.75$ So f(14.75) = 0.0608 These are the required iterations

EXAMPLE

Explain why the equation $e^{-x} = x$ has a solution on the interval [0,1]. Use bisection to find the root to 4 decimal places. Can you prove that there are no other roots?

SOLUTION

If $f(x) = e^{-x} - x$, then $f(0) = 1$, $f(1) = 1/e - 1 < 0$, and hence a root is guaranteed by the Intermediate Value Theorem. Using Bisection, the value of the root is $x^? = .5671$.

Since $f^0(x) = -e^{-x} - 1 < 0$ for all x, the function is strictly decreasing, and so its graph can only cross the x axis at a single point, which is the root.

FALSE POSITION METHOD

This method also known as REGULA FALSI METHOD,, CHORD METHOD ,, LINEAR INTERPOLATION and method is one of the bracketing methods and based on intermediate value theorem.

This method is different from bisection method.

Like the bisection method we are not taking the mid-point of the given interval to determine the next interval and converge faster than bisection method.

ALGORITHM

Given a function f(x) continuous on an interval $[a_0,b_0]$ and satisfying $f(a_0).f(b_0)<0$ for all n = 0,1,2,3………….. then Use following formula to next root

$x_r = x_f - \frac{x_f - x_i}{f(x_f) - f(x_i)} f(x_f)$ **We can also use $x_r = x_{n+1}$,,, $x_f = x_n$,,, $x_i = x_{n-1}$**

STOPING CRITERIA

1. **Interval small enough.**
2. **$|f(c_n)|$ almost zero**
3. **Maximum number of iteration reached**
4. **Same answer.**
5. **Any combination of previous ones**

EXAMPLE

Using Regula Falsi method Solve x^3-9x+1 for roots between x=2 and x=4

SOLUTION

X	2	4
f(x)	-9	29

Since f(2).f(4)<0 therefore root lies between 2 and 4

Using formula

$x_r = x_f - \frac{x_f - x_i}{f(x_f) - f(x_i)} f(x_f)$

For interval [2,4] we have $x_r = 4 - \frac{4-2}{29-(-9)} \times 29 = 2.4737$

Which implies $f(2.4737) = -6.1263$ **(-ve)**

Similarly, other terms are given below

Interval	x_r	$F(x_r)$
[2.4737,4]	2.7399	-3.0905
[2.7399,4]	2.8613	-1.326
[2.8613,4]	2.9111	-0.5298
[2.9111,4]	2.9306	-0.2062
[2.9306,4]	2.9382	-0.0783
[2.9382,4]	2.9412	-0.0275
[2.9412,4]	2.9422	-0.0105
[2.9422,4]	2.9426	-0.0037
[2.9426,4]	2.9439	0.0183
[2.9426,2.9439]	2.9428	-0.0003
[2.9426,2.9439]	2.9428	-0.0003

EXAMPLE

Using Regula Falsi method to find root of equation "$log x - cos x = 0$" upto four decimal places, after 3 successive approximations.

SOLUTION

X	0	1	2
F(X)	-∞	-0.5403	1.1093

Since f(1).f(2)<0 therefore root lies between 1 and 2

Using formula

Xr= x_f - $\frac{x_f - x_i}{f(x_f) - f(x_i)}$ f(x_f)

For interval [1,2] we have x_r=2- $\frac{2-1}{1.1093-(-0.5403)}$ $\times$1.1093=1.3275

Which implies f(2.4737)=0.0424(+ve)

Similarly, other terms are given below

Interval	x_r	F(x_r)
[1,1.3275]	1.3037	0.0013
[1,1.3037]	1.3030	0.0001

Hence the root is 1.3030

KEEP IN MIND

- **Calculate this equation in Radian mod**
- **If you have "log" then use "natural log". If you have "log_{10}" then use "simple log".**

GENERAL FORMULA FOR REGULA FALSI USING LINE EQUATION

Equation of line is

$$\frac{y - f(x_n)}{x - x_n} = \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

Put (x,0) i.e. y=0

$$\frac{-f(x_n)}{x - x_n} = \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

$$\frac{-f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n}$$

$$\frac{-(x_{n-1} - x_n)f(x_n)}{f(x_{n-1}) - f(x_n)} = x - x_n$$

$$x = x_n - \frac{(x_{n-1} - x_n)f(x_n)}{f(x_{n-1}) - f(x_n)}$$

Hence first approximation to the root of f(x) =0 is given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

We observe that $f(x_{n-1})$, $f(x_{n+1})$ are of opposite sign so, we can apply the above procedure to successive approximations.

SECANT METHOD

The secant method is a simple variant of the method of false position which it is no longer required that the function "f" has opposite signs at the end points of each interval generated, not even the initial interval.

In other words, one starts with two arbitrary initial approximations $x_0 \neq x_1$ and continues with

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})} \qquad ; n=1,2,3,4\ldots\ldots\ldots\ldots$$

This method also known as QUASI NEWTON'S METHOD.

ADVANTAGES

1. **No computations of derivatives**
2. **One f(x) computation each step**
3. **Also rapid convergence than Falsi method**

Example 1 Use Secant method to find the root of the function f(x) = cosx + 2sinx + x^2 to 5 decimal places. Don't forget to adjust your calculator for "radians".

Solution: **A closed form solution for x does not exist so we must use a numerical technique. The Secant method is given using the iterative equation:**

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right], \qquad (1)$$

We will use $x_0 = 0$ and $x_1 = -0.1$ as our initial approximations and substituting in (1), we have $x_{n+1} = -0.1 - 0.80533 * \left[\frac{-0.1}{0.80533-1}\right] =$ **−0.51369. The continued iterations can be computed as shown in Table 1 which shows a stop at iteration no. 5 since the error is $x_5 - x_4 < 10^{-5}$ resulting in a root of $x^* = -0.65926$**

Table 1: Iterations for Example-1

Iteration no.	x_{n-1}	x_n	x_{n+1} using (1)	$f(x_{n+1})$	$x_{n+1} - x_n$
1	$x_0 = 0$	$x_1 = -0.1$	-0.51369	0.15203	-0.41369
2	-0.1	-0.51369	-0.60996	0.04605	-0.09627
3	-0.51369	-0.60996	-0.65179	6.60859×10^{-3}	-0.04183
4	-0.60996	-0.65179	-0.65880	4.08003×10^{-4}	-0.00701
5	-0.65179	-0.65880	-0.65926	5.28942×10^{-6}	-0.00046

Example-2: Use Secant method to find the root of the function $f(x) = x^3 - 4$ to 5 decimal places.

Solution Since the Secant method is given using the iterative equation in (1). Starting with an initial value $x_0 = 1$ and $x_1 = 1.5$, using (1) we can compute

$$x_2 = 1.5 - (-0.625)\left[\frac{1.5-1}{-0.625-(-3)}\right] = 1.$$

63158. The continued iterations can be computed as shown in Table 2 which shows a stop at iteration no. 5 since the error is $x_5 - x_4 < 10^{-5}$ resulting in a root of $x^* = 1.58740$,

Table 2: Iterations for Example-2

Iteration no.	x_{n-1}	x_n	x_{n+1} using (1)	$f(x_{n+1})$	$x_{n+1} - x_n$
1	$x_0 = 1$	$x_1 = 1.5$	1.63158	0.34335	0.13158
2	1.5	1.63158	1.58493	-0.01865	-0.04665
3	1.63158	1.58493	1.58733	-0.00054	0.0024
4	1.58493	1.58733	1.58740	-7.95238×10^{-6}	0.00007
5	1.58733	1.58740	1.58740	-7.95238×10^{-6}	< 10−5

Example-3: Use Secant method to find the root of the function $f(x) = 3x + \text{Sin}x - e^x$ to 5 decimal places. Use $x_0 = 0$ and $x_1 = 1$.

Solution

Using (1) we can compute

$$x_2 = 1-(1.12319)\left[\frac{1-0}{1.12319-(-1)}\right] = 0.47099.$$

The continued iterations can be computed as shown in Table 3 which shows a stop at iteration no. 6 since the error is $x_6 - x_5 < 10^{-5}$ resulting in a root of $x^* = 0.36042$

Table 3: Iterations for Example-3

Iteration no.	x_{n-1}	x_n	x_{n+1} using (1)	$f(x_{n+1})$	$x_{n+1} - x_n$
1	$x_0 = 0$	$x_1 = 1$	0.47099	0.26516	-0.52901
2	1	0.47099	0.30751	-0.13482	-0.16348
3	0.47099	0.30751	0.36261	5.47043×10^{-3}	0.0551
4	0.30751	0.36261	0.36046	9.58108×100^{-5}	-0.00215
5	0.36261	0.36046	0.36042	-4.26049×10^{-6}	-0.00004
6	0.36046	0.36042	0.36042	-4.26049×10^{-6}	< 10−5

Example-4: Solve the equation exp(−x) = 3log(x) to 5 decimal places using secant method, assuming initial guess $x_0 = 1$ and $x_1 = 2$.

Solution

Let f(x) = exp(−x) − 3log(x), to solve the given, it is now equivalent to find the root of f(x). Using (1) we can compute

$x_2 = 2 - (-0.76775)\left[\frac{2-1}{-0.76775-(0.36788)}\right] = 1.32394$**. The continued iterations can be computed as shown in Table 4 which shows a stop at iteration no. 5 since the error is $x_5 - x_4 < 10^{-5}$ resulting in a root of $x^* = 1.24682$,**

Table 4: Iterations for Example-4

Iteration no.	x_{n-1}	x_n	x_{n+1} using (1)	$f(x_{n+1})$	$x_{n+1} - x_n$
1	$x_0 = 1$	$x_1 = 2$	1.32394	-0.09952	-0.67606
2	2	1.32394	1.22325	0.03173	-0.10069
3	1.32394	1.22325	1.24759	-1.01955×10^{-3}	0.02434
4	1.22325	1.24759	1.24683	-7.27178×10^{-6}	-0.00076
5	1.24759	1.24683	1.24682	6.05199×10^{-6}	< 10−5

FIXED POINT

The real number "x" is a fixed point of the function "f" if f(x) =x

The number x=0.7390851332 is an approximate fixed point of f(x) = cosx

REMARK

Fixed point are roughly divided into three classes

ASYMPTOTICALLY STABLE: with the property that all nearby solutions converge to it.

STABLE: All nearby solutions stay nearby.

UNSTABLE: Almost all of whose nearby solutions diverge away from the fixed point

FIXED POINT ITERATION METHOD

ALGORITHM

1. **Consider f(x) =0 and transform it to the form x= φ (x)**
2. **Choose an arbitrary x_0**
3. **Do the iterations $x_{k+1}=\varphi(x_k)$; k=0,1,2,3……….**

STOPING CRITERIA

Let "ϵ" be the tolerance value

1. $|x_k - x_{k-1}| \le \epsilon$
2. $|x_k - f(x_k)| \le \epsilon$
3. **Maximum number of iterations reached.**
4. **Any combination of above.**

CONVERGENCE CRITERIA

Let "x" be exact root such that r=f(x) out iteration is $x_{n+1} = f(x_n)$

Define the error $\epsilon_n = x_n - r$ Then

$$\epsilon_{n+1} = x_{n+1} - r = f(x_n) - r = f(x_n) - f(r) = f'(\S)(x_n - r)$$

(Where § ∈ (x_n, r) ; since f is continuous)

$$\epsilon_{n+1} = f'(\S)\epsilon_n \Rightarrow \epsilon_{n+1} \le |f'(\S)||\epsilon_n|$$

OBSERVATIONS

If $|f'(\S)| < 1$, error decreases, the iteration converges (linear convergence)

If $|f'(\S)| \ge 1$, error increases, the iteration diverges.

REMEMBER: If $|\varphi'(x)| < 1$ in questions then take that point as initial guess.

EXAMPLE

Find the root of equation $2x = cosx + 3$ correct to three decimal points using fixed point iteration method.

SOLUTION

Given that $f(x) = 2x - cosx - 3 = 0$

X	0	1	2
F(X)	-4	-1.5403	1.4161

Root lies between "1" and "2"

Now $2x - cosx - 3 = 0 \Rightarrow x = \frac{cosx+3}{2} = \varphi(x)$

$\Rightarrow \varphi'(x) = \frac{1}{2}(-sinx) \Rightarrow |\varphi'(x)| = |\frac{1}{2}(-sinx)|$

Now $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{1}{2}(cox x_n + 3)$

If by putting 1 we get $|\varphi'(x)| < 1$ then take it as "x_0" if not then check for 2 rather take their mid-point

Here we will take "x_0" as mid-point. So

$X_0 = \frac{1+2}{2} = 1.5$

$X_1 = \frac{1}{2}(cosx_0 + 3) = 1.5354$	$F(x_1) = 0.0354$
$X_2 = \frac{1}{2}(cosx_1 + 3) = 1.5177$	$F(x_2) = -0.0177$
$X_3 = 1.5265$	$F(x_3) = 0.0087$
$x_4 = 1.5221$	$F(x_4) = -0.0045$
$X_5 = 1.5243$	$F(x_5) = 0.0021$
$X_6 = 1.5232$	$F(x_6) = -0.0012$
$X_7 = 1.5238$	$F(x_7) = 0.0006$
$X_8 = 1.5235$	$F(x_8) = -0.0003$
$X_9 = 1.5236$	$F(x_9) = 0.0000$

Hence the real root is 1.5236

EXAMPLE

Find the root of equation $e^{-x} = 10x$ correct to four decimal points using fixed point iteration method.

SOLUTION

Given that

$$f(x) = e^{-x} - 10x = 0$$

X	0	1
F(X)	1	-9.6321

Root lies between "0" and "1"

Now $e^{-x} - 10x = 0 \Rightarrow x = \frac{e^{-x}}{10} = \varphi(x)$

$$\Rightarrow \varphi'(x) = -\frac{e^{-x}}{10}$$

Now since $|\varphi'(0)| = 0.1$ is less than "1" therefore $x_0 = 0$

Now $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{e^{-x_n}}{10}$

$X_1 = \frac{e^{-x_0}}{10} = \frac{e^{-0}}{10}$ = 0.1000	$F(x_1)$ = -0.0952
X_2= 0.0905	$F(x_2)$ = 0.0085
X_3 = 0.0913	$F(x_3)$ = -0.0003
x_4 = 0.0913	$F(x_4)$ = -0.0003

Hence the real root is 0.0913

NEWTON RAPHSON METHOD

Nature and Nature's laws lay hid in night:
God said, Let Newton be! And all was light.
Alexander Pope, 1727

The Newton Raphson method is a powerful technique for solving equations numerically. It is based on the idea of linear approximation. Usually converges much faster than the linearly convergent methods.

ALGORITHM

The steps of Newton Raphson method to find the root of an equation "f(x) =0" are

Evaluate $f'(x)$

Use an initial guess (value on which f(x) and $f''(x)$ becomes (+ve) of the roots "x_n" to estimate the new value of the root "x_{n+1}" as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \; \ldots\ldots\ldots\ldots \textit{this value is known as Newton's iteration}$$

STOPING CRITERIA

1. **Find the absolute relative approximate error as** $|\epsilon_\propto| = |\frac{x_{n+1}-x_n}{x_{n+1}}| \times 100$
2. **Compare the absolute error with the pre-specified relative error tolerance "ϵ_s".**
3. **If $|\epsilon_a| > \epsilon_s$ then go to next approximation. Else stop the algorithm.**
4. **Maximum number of iterations reached.**
5. **Repeated answer.**

CONVERGENCE CRITERIA

Newton method will generate a sequence of numbers (x_n) ; n$\geq$ 0, that converges to the zero "x_*" of "f" if

- **"f" is continuous.**
- **"x_*" is a simple zero of "f".**
- **"x_0" is close enough to "x_*"**

When the Generalized Newton Raphson method for solving equations is helpful?

To find the root of "f(x)=0" with multiplicity "p" the Generalized Newton formula is required.

What is the importance of Secant method over Newton Raphson method?

Newton Raphson method requires the evaluation of derivatives of the function and this is not always possible, particularly in the case of functions arising in practical problems.

In such situations Secant method helps to solve the equation with an approximation to the derivatives.

Why Newton Raphson method is called Method of Tangent?

In this method we draw tangent line to the point" $P_0(x_0,f(x_0))$". The (x,0) where this tangent line meets x-axis is 1^{st} approximation to the root.

Similarly, we obtained other approximations by tangent line. So, method also called Tangent method.

Difference between Newton Raphson method and Secant method.

Secant method needs two approximations x_0,x_1 to start, whereas Newton Raphson method just needs one approximation i.e. x_0

Newton Raphson method converges faster than Secant method.

Newton Raphson method is an Open method, how?

Newton Raphson method is an open method because initial guess of the root that is needed to get the iterative method started is a single point. While other open methods use two initial guesses of the root but they do not have to bracket the root.

INFLECTION POINT

For a function "f(x)" the point where the concavity changes from up-to-down

or down-to-up is called its Inflection point.

e.g. $f(x) = (x-1)^3$ changes concavity at x=1,, Hence (1,0) is an Inflection point.

DRAWBACKS OF NEWTON'S RAPHSON METHOD

- **Method diverges at inflection point.**
- **For f(x)=0 Newton Raphson method reduce. So one must be avoid division by zero. Rather method not converges.**
- **Root jumping is another drawback.**
- **Results obtained from Newton Raphson method may oscillate about the Local Maximum or Minimum without converging on a root but converging on the Local Maximum or minimum.**

Eventually, it may lead to division by a number close to zero and may diverge.

- **The requirement of finding the value of the derivatives of f(x) at each approximation is either extremely difficult (if not possible) or time consuming.**

FORMULA DARIVATION FOR NR-METHOD

Given an equation "f(x) = 0" suppose "x_0" is an approximate root of "f(x) = 0"

Let $x_1 = x_0 + h \dots\dots\dots\dots\dots\dots (1)$ $\qquad$ $since\ \ x_1 - x_0 = h$

Where "h" is the small; exact root of f(x)=0

Then $f(x_1) = 0 = f(x_0 + h)$ $\qquad$ $since\ \ x_1 = x_0 + h$

By Taylor theorem

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) \dots\dots\dots\dots\dots\dots. = 0$$

Since "h" is small therefore neglecting higher terms we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

$$(1) \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots = \vdots - \quad \vdots$$

$$\vdots = \vdots - \quad \vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is required Newton's Raphson Formula.

EXAMPLE

Apply Newton's Raphson method for $cosx = xe^x \; at \; x_0 = 1$ correct to three decimal places.

SOLUTION

$f(x) = cosx - xe^x$

$f'(x) = -sinx - e^x - xe^x$

Using formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

at $x_0 = 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.653 \quad (after\ solving)$

$f(x_1) = -0.460 \; ; \; f'(x_1) = -3.783$

Similarly

n	x_n	$f(x_n)$	$f'(x_n)$
2	0.531	-0.041	-3.110
3	0.518	-0.001	-3.043
4	0.518	-0.001	-3.043

Hence root is "0.518"

REMARK

1. **If two are more roots are nearly equal, then method is not fastly convergent.**
2. **If root is very near to maximum or minimum value of the function at the point, NR-method fails.**

EXAMPLE

Apply Newton's Raphson method for $xlog_{10}x = 4.77$ correct to two decimal places.

SOLUTION

$$f(x) = xlog_{10}x - 4.77$$

$$f'(x) = log_{10}x + x\frac{1}{x}log_{10}e$$

$$f'(x) = log_{10}x + log_{10}e$$

$$f'(x) = log_{10}x + 0.4343 \qquad since\ e = 2.71828$$

$$f''(x) = \frac{1}{x}log_{10}e = \frac{0.4343}{x}$$

For interval

X	0	1	2	3	4	5	6	7
f(x)	-4.77	-4.77	-4.17	-3.34	-2.36	-1.28	-0.10	1.15

Root lies between 6 and 7 and let x_0 =7

Using formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Thus

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 6.10 \quad after\ solving$$

$$f(x_1) = 0.02\ ;\ f'(x_1) = 1.22$$

Similarly

n	x_n	$f(x_n)$	$f'(x_n)$
2	6.08	0.00	0.00

Hence root is "6.08"

GEOMETRICAL INTERPRETATION (GRAPHICS) OF NEWTON RAPHSON FORMULA

Suppose the graph of function "y=f(x)" crosses x-axis at " $\propto$ " then "$x = \propto$ " is the root of equation"$f(x) = 0$".

CONDITION

Choose "x_0" such that "$f(x)$" and f"(x) have same sign. If "$(x_0, f(x_0))$" is a point then slope of tangent at "$(x_0, f(x_0)) = m = \frac{dy}{dx}|_{(x_0, f(x_0))} = f'(x_0)$"

Now equation of tangent is

$$y - y_0 = m(x - x_0)$$

$$y - f(x_0) = f'(x_0)(x - x_0) \qquad \text{.. (i)}$$

Since $(x_1, f(x_1) = y_1 = 0)$ as we take x_1 as exact root

(i)⟹ $$0 - f(x_0) = f'(x_0)(x - x_0)$$

$$-\frac{f(x_0)}{f\prime(x0)} = x_1 - x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)}$$

Which is first approximation to the root " $\propto$ " . If "P_1" is a point on the curve corresponding to "x_1" then tangent at "P_1" cuts x-axis at $P_1(x_2, 0)$ which is still closer to "$\propto$" than "x_1". Therefore "x_2" is a 2^{nd} approximation to the root.

Continuing this process, we arrive at the root "$\propto$".

CONDITION FOR CONVERGENCE OF NR-METHOD

Since by Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

And by General Iterative formula

$$x_{n+1} = \varphi(x_n) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Comparing (1) and (2)

$$\varphi(x_n) = \quad x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\varphi(x) = \quad x - \frac{f(x)}{f'(x)} \qquad (simply)$$

Since by iterative method condition for convergence is

$$|\varphi'(x)| < 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

So

$$\varphi'(x) = 1 - [\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}]$$

$$\varphi'(x) = 1 - \frac{(f'(x))^2}{(f'(x))^2} + \frac{f(x)f''(x)}{(f'(x))^2}$$

$\varphi'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$ **Using in (3) we get**

$$\left|\frac{f(x)f''(x)}{\left(f'(x)\right)^2}\right| < 1 \qquad \Rightarrow |f(x)f''(x)| < \left(f'(x)\right)^2$$

Which is required condition for convergence of Newton Raphson method, provided that initial approximation "x_0" is choose sufficiently close to the root and $f(x), f'(x), f''(x)$ are continuous and bounded in any small interval containing the root.

NEWTON RAPHSON METHOD IS QUADRATICALLY CONVERGENT

(OR)

NEWTON RAPHSON METHOD HAS SECOND ORDER CONVERGENCE

(OR)

ERROR FOR NEWTON RAPHSON METHOD

Let " $\propto$ " be the root of f(x) =0 and

$$\begin{pmatrix} x_n - \propto = \epsilon_n \\ x_{n+1} - \propto = \epsilon_{n+1} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

If we can prove that $\epsilon_{n+1} = k\,\epsilon_n^p$ **where "k" is constant then "p" is called order of convergence of iterative method then we are done.**

Since by Newton Raphson formula we have

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ **Then using (1) in it**

$$\propto + \epsilon_{n+1} = \propto + \epsilon_n - \frac{f(\propto + \epsilon_n)}{f'(\propto + \epsilon_n)}$$

$$\epsilon_{n+1} = \epsilon_n - \frac{f(\propto + \epsilon_n)}{f'(\propto + \epsilon_n)}$$

Since by Taylor expansion we have

$$\epsilon_{n+1} = \epsilon_n - \frac{f(\propto) + \epsilon_n f'(\propto) + \frac{\epsilon_n^2}{2!} f''(\propto)}{f'(\propto) + \epsilon_n f''(\propto) + \frac{\epsilon_n^2}{2!} f'''(\propto)}$$

Since " $\propto$ " is root of f(x) therefore "$f(\propto) = 0$"

$$\epsilon_{n+1} = \epsilon_n - \frac{\epsilon_n f'(\propto) + \frac{\epsilon_n^2}{2!} f''(\propto)}{f'(\propto) + \epsilon_n f''(\propto) + \frac{\epsilon_n^2}{2!} f'''(\propto)}$$

$$\epsilon_{n+1} = \frac{\epsilon_n \, [f'(\propto) + \epsilon_n \, f''(\propto) + \frac{\epsilon_n^2}{2!} f'''(\propto)] - \epsilon_n \, f'(\propto) - \frac{\epsilon_n^2}{2!} f''(\propto)}{f'(\propto) + \epsilon_n \, f''(\propto) + \frac{\epsilon_n^2}{2!} f'''(\propto)}$$

$$\epsilon_{n+1} = \frac{\frac{\epsilon_n^2}{2} f''(\propto) + \frac{\epsilon_n^3}{2!} f'''(\propto)}{f'(\propto) + \epsilon_n \, f''(\propto) + \frac{\epsilon_n^2}{2!} f'''(\propto)}$$

After neglecting higher terms

$$\epsilon_{n+1} = \frac{\frac{\epsilon_n^2}{2} f''(\propto)}{f'(\propto) + \epsilon_n \, f''(\propto)}$$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 \, f''(\propto)}{2f'(\propto)[1 + \frac{\epsilon_n \, f''(\propto)}{f'(\propto)}]}$$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 \, f''(\propto)}{2f'(\propto)} \, [1 + \frac{\epsilon_n \, f''(\propto)}{f'(\propto)}]^{-1}$$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 \, f''(\propto)}{2f'(\propto)} [1 - (-1) \frac{\epsilon_n \, f''(\propto)}{f'(\propto)} + \boldsymbol{neglected}$$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 \, f''(\propto)}{2f'(\propto)} - \frac{\epsilon_n^3 \, f''^2(\propto)}{2f'^2(\propto)}$$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 \, f''(\propto)}{2f'(\propto)} = k \, \epsilon_n^2 \qquad \boldsymbol{where} \;\; k = \frac{f''(\propto)}{2f'(\propto)}$$

It shows that Newton Raphson method has second order convergence

Or

Converges quadratically.

Convergence of Newton-Raphson Method

- Usually converges quadratically

Example: $f(x) = e^{-x} - x$ (true solution = 0.567143290409784)

Solved with 2 methods:

Newton-Raphson with $x_0=0$

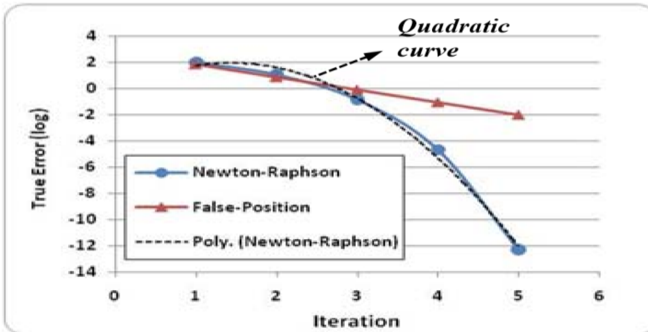
False-Position Method with $x_l=0$ and $x_u=20$

Newton-Raphson

	Iterations	true error
x_0 =	0	100.000000000%
x_1 =	0.500000000000000	11.838858282%
x_2 =	0.566311003197218	0.146750782%
x_3 =	0.567143165034862	0.000022106%
x_4 =	0.567143290409781	0.000000000%

False-Position

	Iterations	true error
x_0 =	0.952380952	67.925984240%
x_1 =	0.607944265065116	7.194121018%
x_2 =	0.571658116501746	0.796064446%
x_3 =	0.567645088312370	0.088478152%
x_4 =	0.567199089558233	0.009838633%



EXAMPLE 2.10 Newton's Method for a Problem with a Root of Multiplicity > 1

Consider the function $f(x) = x(1 - \cos x)$, which has a root of multiplicity three at $x = 0$. The following table shows the results of ten iterations of Newton's method applied to this problem with a starting value of $p_0 = 1$. For comparison, the results of the bisection method, starting from the interval $[-2, 1]$ are shown in the third column.

	Newton's Method	Bisection Method
1	0.6467039965	−0.5000000000
2	0.4259712109	0.2500000000
3	0.2825304410	−0.1250000000
4	0.1879335654	0.0625000000
5	0.1251658102	−0.0312500000
6	0.0834075192	0.0156250000
7	0.0555942620	−0.0078125000
8	0.0370596587	0.0039062500
9	0.0247054965	−0.0019531250
10	0.0164700517	0.0009765625

NEWTON RAPHSON EXTENDED FORMULA

(CHEBYSHEVES FORMULA OF 3RD ORDER)

Consider f(x) =0. Expand f(x) by Taylor series in the neighborhood of "x_0". We obtain after retaining the first term only.

$$0 = f(x) = f(x_0) + (x - x_0)\, f'(x_0) + neglected \Rightarrow 0 = f(x_0) + (x - x_0)\, f'(x_0)$$

$$\Rightarrow x - x_0 = -\frac{f(x_0)}{f'(x_0)} \quad \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This the first approximation to the root therefore

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots..(1)$$

Again expanding f(x) by Taylor Series and retaining the second order term only

$$0 = f(x) = f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0)$$

$$0 = f(x_1) = f(x_0) + (x_1 - x_0)\, f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) \qquad \therefore f(x) = f(x_1)$$

$$0 = f(x_0) + (x_1 - x_0)\, f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) \quad \dots\dots\dots\dots\dots\dots\dots..(2)$$

Using eq. (1) in (2) we get

$$f(x_0) + (x_1 - x_0)\, f'(x_0) + \frac{1}{2}[-\frac{f(x_0)}{f'(x_0)}]^2\, f''(x_0) = 0$$

$$f(x_0) + (x_1 - x_0)\, f'(x_0) = -\frac{1}{2}\left[\frac{f(x_0)}{f'(x_0)}\right]^2\, f''(x_0)$$

$$x_1\, f'(x_0) = x_0\, f'(x_0) - f(x_0) - \frac{1}{2}[\frac{f(x_0)}{f'(x_0)}]^2\, f''(x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3}\, f''(x_0)$$

This is Newton Raphson Extended formula. Also known as "Chebysheves formula of third order"

NEWTON SCHEME OF ITERATION FOR FINDING THE SQUARE ROOT OF POSITION NUMBER

The square root of "N" can be carried out as a root of the equation

$x = \sqrt{N} \implies x^2 = N \implies x^2 - N = 0$

Here $f(x) = x^2 - N$; $f(x_n) = x_n^2 - N$

$f'(x) = 2x$; $f'(x_n) = 2x_n$

Using Newton Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\implies x_{n+1} = x_n - \frac{(x_{n-N}^2)}{2x_n}$

$\implies x_{n+1} = \frac{1}{2}[x_n + \frac{N}{x_n}]$ **This is required formula.**

QUESTION

Evaluate $\sqrt{12}$ by Newton Raphson formula.

SOLUTION

Let $x = \sqrt{12} \implies x^2 = 12 \implies x^2 - 12 = 0$

Here $f(x) = x^2 - 12$; $f'(x) = 2x$; $f''(x) = 2$

X	0	1	2	3	4
F(x)	-12	-11	-8	-3	4

Root lies between 3 and 4 and x_0 =4

Now using formula $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right] \implies x_{n+1} = \frac{1}{2}\left[x_n + \frac{12}{x_n}\right]$ ………………….. (1)

For n=0 $x_1 = \frac{1}{2}\left[x_0 + \frac{12}{x_0}\right] \implies x_1 = \frac{1}{2}\left[4 + \frac{12}{4}\right]$=3.5

For n=2 $x_2 = \frac{1}{2}\left[x_1 + \frac{12}{x_1}\right] \implies x_2 = \frac{1}{2}\left[3.5 + \frac{12}{3.5}\right] = 3.4643$

Similarly $x_3 = 3.4641$ *and* $x_4 = 3.4641$

Hence $\sqrt{12} = 3.4641$

NEWTON SCHEME OF ITERATION FOR FINDING THE "pth" ROOT OF POSITION NUMBER "N"

Consider $x = N^{\frac{1}{p}} \quad \Rightarrow \quad x^p = N \Rightarrow x^p - N = 0$

Here $\quad f(x) = x^p - N \quad ; \quad f(x_n) = x_n^p - N$

$\quad f'(x) = px^{p-1} \quad ; \quad f'(x_n) = px_n^{p-1}$

Since by Newton Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \Rightarrow x_{n+1} = x_n - \frac{(x_n^p - N)}{\left(px_n^{p-1}\right)} \quad \Rightarrow x_{n+1} = \frac{1}{px_n^{p-1}}\left[px_n^{p-1+1} - x_n^p + N\right]$$

$$x_{n+1} = \frac{1}{px_n^{p-1}}\left[(p-1)x_n^p + N\right] \Rightarrow x_{n+1} = \frac{1}{p}\left[\frac{(p-1)x_n^p + N}{x_n^{p-1}}\right]$$ Required formula for pth root.

QUESTION

Obtain the cube root of 12 using Newton Raphson iteration.

SOLUTION

Consider $x = 12^{\frac{1}{3}} \quad \Rightarrow \quad x^3 = 12 \Rightarrow x^3 - 12 = 0$

Here $\quad f(x) = x^3 - 12$ and $f'(x) = 3x^2 \quad ; \quad f''(x) = 6x$

For interval

X	0	1	2	3
F(x)	-12	-11	-4	15

Root lies between 2 and 3 and x_0=3

Since by Newton Raphson formula for pth root.

$$x_{n+1} = \frac{1}{p}\left[\frac{(p-1)x_n^p + N}{x_n^{p-1}}\right] \quad \Rightarrow x_{n+1} = \frac{1}{3}\left[\frac{(3-1)x_n^3 + 12}{x_n^{3-1}}\right] = \frac{1}{3}\left[\frac{2x_n^3 + 12}{x_n^2}\right]$$

Put n=0 $\quad x_1 = \frac{1}{3}\left[\frac{2x_0^3 + 12}{x_0^2}\right] = \frac{1}{3}\left[\frac{2(3)^3 + 12}{(3)^2}\right] = 2.4444$

Similarly

$x_2 = 2.2990$, $x_3 = 2.2895$, $x_4 = 2.2894 \quad x_5 = 2.2894$

Hence $\quad \sqrt[3]{12} = 2.2894$

DARIVATION OF NEWTON RAPHSON METHOD FROM TAYLOR SERIES

Newton Raphson method can also be derived from Taylor series.

For the general function “f(x)” Taylor series is

$$f(x_{n+1}) = f(x_n) + f'(x_{n+1} - x_n) + \frac{f''(x_n)}{2!}(x_{n+1} - x_n)^2 + \cdots \ldots \ldots \ldots \ldots \ldots$$

As an approximation, taking only the first two terms of the R.H.S.

$$f(x_{n+1}) = f(x_n) + f'(x_{n+1} - x_n)$$

And we are seeking a point where f(x) =0

That is If we assume f (x_{n+1}) =0

$$\Longrightarrow \quad f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0$$

This gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the formula for Newton Raphson Method.

THE SOLUTION OF LINEAR SYSTEM OF EQUATIONS

A system of "m" linear equations in "n" unknowns "$x_1, x_2, x_3, \dots\dots\dots.., x_n$" is a set of the equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \quad \dots\dots\dots\dots\dots\dots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \quad \dots\dots\dots\dots\dots\dots a_{2n}x_n = b_2$$

…………………………………………………………………………………….

…………………………………………………………………………………….

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \quad \dots\dots\dots\dots\dots\dots a_{mn}x_n = b_m$$

Where the coefficients "a_{ik}" and "b_i" are given numbers.

The system is said to be homogeneous if all the "b_i" are zero. Otherwise it is said to be non-homogeneous.

SOLUTION OF LINEAR SYSTEM EQUATIONS

A solution of system is a set of numbers "$x_1, x_2, x_3, \dots\dots\dots.., x_n$" which satisfy all the "m" equations.

PIVOTING: **Changing the order of equations is called pivoting.**

We are interested in following types of Pivoting

1. **PARTIAL PIVOTING**
2. **TOTAL PIVOTING**

PARTIAL PIVOTING

In partial pivoting we interchange rows where pivotal element is zero.

In Partial Pivoting if the pivotal coefficient "a_{ii}" happens to be zero or near to zero, the i^{th} column elements are searched for the numerically largest element. Let the j^{th} row (j>i) contains this element, then we interchange the "i^{th}" equation with the "j^{th}" equation and proceed for elimination. This process is continued whenever pivotal coefficients become zero during elimination.

TOTAL PIVOTING

In Full (complete, total) pivoting we interchange rows as well as column.

In Total Pivoting we look for an absolutely largest coefficient in the entire system and start the elimination with the corresponding variable, using this coefficient as the pivotal coefficient (may change row and column). Similarly, in the further steps. It is more complicated than Partial Pivoting. Partial Pivoting is preferred for hand calculation.

<u>Why is Pivoting important?</u>

Because Pivoting made the difference between non-sense and a perfect result.

PIVOTAL COEFFICIENT

For elimination methods (Guass's Elimination, Guass's Jordan) the coefficient of the first unknown in the first equation is called Pivotal Coefficient.

BACK SUBSTITUTION

The analogous algorithm for upper triangular system "Ax=b" of the form

$$\begin{pmatrix} a_{11} & a_{12} \dots\dots\dots .. a_{1n} \\ 0 & a_{22} \dots\dots\dots .. a_{2n} \\ \vdots & \vdots \qquad\qquad \vdots \\ 0 & 0 \quad \cdots\cdots \quad a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Is called Back Substitution.

The solution "x_i" is computed by $x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$ $\quad ; i = 1, 2, 3, \dots\dots\dots n$

FORWARD SUBSTITUTION

The analogous algorithm for lower triangular system "Lx=b" of the form

$$\begin{pmatrix} l_{11} & 0 \quad \dots\dots .0 \\ l_{21} & l_{22} \dots\dots\dots .. 0 \\ \vdots & \vdots \qquad\qquad \vdots \\ l_{n1} & l_{n2} \cdots\cdots \quad l_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Is called Forward Substitution.

The solution "x_i" is computed by $x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} x_j}{l_{ii}}$ $\quad ; i = 1, 2, 3, \dots\dots\dots n$

THINGS TO REMEMBER

Let the system $AX = B$ is given

- **If $B \neq 0$ then system is called non homogenous system of linear equation.**
- **If $B = 0$ then $AX = 0$ then system is called homogenous system of linear equation.**
- **If the system $AX = B$ has solution then this system is called consistent.**
- **If the system $AX = B$ has no solution then this system is called inconsistent.**

RANK OF A MATRIX

The rank of a matrix 'A' is equal to the number of non – zero rows in its echelon form or the order of I_r in the conical form of A.

KEEP IN MIND

- **TYPE I: when number of equations is equal to the number of variables and the system $AX = B$ is non – homogeneous then unique solution of the system exists if matrix 'A' is non- singular after applying row operation.**
- **TYPE II: when number of equations is not equal (may be equal) to the number of variables and the system $AX = B$ is non – homogeneous then system has a solution if $rank\ A = rankA_b$**
- **TYPE III: a system of 'm' homogeneous linear equations $AX = 0$ in 'n' unknown has a non- trivial solution if $rank\ A < n$ where 'n' is number of columns of A.**
- **TYPE IV: if $rank\ A = rankA_b < number\ of\ unknown$ then infinite solution exists**
- **TYPE V: if $rank\ A \neq rankA_b$ then no solution exists**

GUASS ELIMINATION METHOD

ALGORITHM

- **In the first stage, the given system of equations is reduced to an equivalent upper triangular form using elementary transformation.**
- **In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order "$x_n, x_{n-1}, \ldots \ldots \ldots \ldots x_2, x_1$"**

REMARK

Guass's Elimination method fails if any one of the Pivotal coefficient become zero. In such a situation, we rewrite the equation in a different order to avoid zero Pivotal coefficients.

QUESTION **Solve the following system of equations using Elimination Method.**

$$2x + 3y - z = 5$$

$$4x + 4y - 3z = 3$$

$$-2x + 3y - z = 1$$

SOLUTION **We can solve it by elimination of variables by making coefficients same.**

$$2x + 3y - z = 5 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (i)$$

$$4x + 4y - 3z = 3 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (ii)$$

$$-2x + 3y - z = 1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (iii)$$

Multiply (i) by 2 and subtracted by (ii) $\quad 2y + z = 7 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (iv)$

Adding (i) and (iii) $\quad 6y - 2z = 6 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (v)$

Now eliminating "y" Multiply (iv) by 3 then subtract from (v) $\quad z = 3$

Using "z" in (iv) we get $\quad y = 2 \quad$ **and Using "y", "z" in (i) we get** $\quad x = 1$

Hence solution is $\quad x = 1, y = 2, z = 3$

QUESTION

Solve the following system of equations by Guass's Elimination method with partial pivoting.

$$x + y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

SOLUTION

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 7 \\ 16 \end{bmatrix} \quad \sim R_{12} \quad \Rightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 16 \end{bmatrix} \quad \sim \frac{1}{3} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 16 \end{bmatrix} \quad \sim R_2 - R_1 \Rightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 0 \end{bmatrix} \quad \sim R_3 - 2R_1$$

2^{nd} row cannot be used as pivot row as a_{22} =0, So interchanging the 2^{nd} and 3^{rd} row we get

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \sim R_{23}$$

Using back substitution

$$-\frac{1}{3} z = -1 \quad \Rightarrow z = 3$$

$$-y + \frac{1}{3} z = 0 \quad \Rightarrow y = 3 \quad \therefore z = 3$$

$$x + y + \frac{4}{3} z = 8 \quad \Rightarrow x = 3 \quad \therefore y = 3 \;,\; z = 3$$

QUESTION

Solve the following system of equations using Guass's Elimination Method with partial pivoting.

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$

$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$

$$4x_1 + 5x_2 + 65x_3 + 2x_4 = 26$$

$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

SOLUTION

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 26 \\ 21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix} \quad \sim R_{14}$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix} \quad \sim \frac{1}{9} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 4 & 2 & 8 \\ 4 & 5 & 65 & 2 \\ 4 & 10 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 24 \\ 26 \\ 32 \end{bmatrix} \quad \sim R_{24}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 4 & 2 & 8 \\ 0 & \frac{29}{9} & \frac{85}{18} & 2 \\ 0 & \frac{74}{9} & \frac{29}{9} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 24 \\ 16.6668 \\ 22.668 \end{bmatrix} \quad \sim R_3 - 4R_1 \text{ and } \sim R_4 - 4R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 29/9 & 85/18 & 2 \\ 0 & 74/9 & 29/9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ 16.6668 \\ 22.668 \end{bmatrix} \quad \sim \frac{1}{4} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & ½ & 2 \\ 0 & 0 & 3.111 & -4.444 \\ 0 & 0 & -0.889 & 16.444 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ -2.6665 \\ -26.665 \end{bmatrix} \qquad \sim R_3 - \frac{29}{9}R_2 \text{ and } \sim R_4 - \frac{74}{9}R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 1 & -1.428 \\ 0 & 0 & 0 & 15.175 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ -0.857 \\ -27.427 \end{bmatrix} \qquad \sim \frac{R_3}{3.111} \text{ and } \sim R_4 + 0.889R_3$$

$\Rightarrow \quad 15.175x_4 = -27.427$

$\Rightarrow x_4 = -1.8074$

$\Rightarrow x_3 - 1.428x_4 = -0.857$

$\Rightarrow x_3 = -3.438 \quad \therefore x_4 = -1.8074$

$\Rightarrow x_2 + \frac{1}{2}x_3 + 2x_4 = 6$

$\Rightarrow x_2 = 11.3338 \quad \therefore x_4 = -1.8074 \ , \ x_3 = -3.438$

$\Rightarrow x_1 + \frac{4}{9}x_2 + \frac{4}{9}x_3 = 2.333$

$\Rightarrow x_1 = -1.1762 \quad \therefore x_2 = 11.3338 \ , \ x_3 = -3.438$

Hence required solutions are

$x_1 = -1.1762 \ , \ x_2 = 11.3338 \ , \ x_3 = -3.438 \ , \ x_4 = -1.8074$

GUASS JORDAN ELIMINATION METHOD

The method is based on the idea of reducing the given system of equations $"Ax = b"$ to a diagonal system of equations $"Ix = b"$ where $"I"$ is the identity matrix, using row operation. It is the verification of Gauss's Elimination Method.

ALGORITHM

1) **Make the elements below the first pivot in the augmented matrix as zeros, using the elementary row transformation.**
2) **Secondly make the elements below and above the pivot as zeros using elementary row transformation.**
3) **Lastly divide each row by its pivot so that the final matrix is of the form**

$$\begin{bmatrix} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & d_3 \end{bmatrix}$$

Then it is easy to get the solution of the system as $x_1 = d_1, x_2 = d_2, x_3 = d_3$

Partial Pivoting can also be used in the solution. We may also make the pivot as "1" before performing the elimination.

ADVANTAGE/DISADVANTAGE

The Guass's Jordan method looks very elegant as the solution is obtained directly. However, it is computationally more expensive than Guass's Elimination. Hence we do not normally use this method for the solution of the system of equations.

The most important application of this method is to find inverse of a non-singular matrix.

What is Gauss Jordan variation?

In this method Zeroes are generated both below and above each pivot, by further subtractions. The final matrix is thus diagonal rather than triangular and back substitution is eliminated. The idea is attractive but it involves more computing than the original algorithm, so it is little used.

QUESTION

Solve the system of equations using Elimination method

$$x + 2y + z = 8$$

$$2x + 3y + 4z = 20$$

$$4x + 3y + 2z = 16$$

ANSWER

$$\begin{bmatrix} 1 & 2 & 1 & \vdots & 8 \\ 2 & 3 & 4 & \vdots & 20 \\ 4 & 3 & 2 & & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & -1 & -2 & \vdots & 4 \\ 0 & -5 & -2 & & -16 \end{bmatrix} \quad R_2 - R_1 \; and \; R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 1 & -2/5 & & 16/5 \end{bmatrix} \quad (-1)R_2 \; and \; (-1/5)R_3$$

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 0 & 12/5 & & 36/5 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad (5/12)R_3$$

$$\begin{bmatrix} 1 & 2 & 0 & & 5 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad R_1 - R_3 \; and \; R_2 + 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & & 1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad R_1 - 2R_2$$

Hence solutions are $x = 1, y = 2, z = 3$

MATRIX INVERTION

A "$n \times n$" matrix "M" is said to be non-singular (or Invertible) if a "$n \times n$" matrix "M^{-1}" exists with "MM^{-1}" $=$ "$M^{-1}M$" $= I$ then matrix "M^{-1}" is called the inverse of"M". A matrix without an inverse is called Singular (or Non-invertible)

MATRIX INVERSION THROUGH GUASS ELIMINATION

1. Place an identity matrix, whose order is same as given matrix.
2. Convert matrix in upper triangular form.
3. Take largest value as Pivot.
4. Using back substitution get the result.

NOTE: In order to increase the accuracy of the result, it is essential to employ Partial Pivoting. In the first column use absolutely largest coefficient as the pivotal coefficient (for this we have to interchange rows if necessary). Similarly, for the second column and vice versa.

MATRIX INVERSION THROUGH GUASS JORDAN ELIMINATION

1. Place an identity matrix, whose order is same as given matrix.
2. Convert matrix in upper triangular form.
3. No need to take largest value as Pivot.
4. Using back substitution get the result.

QUESTION : Find inverse using Guass Elimination Method $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$

ANSWER

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_{12} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \quad \frac{1}{4}R_1$$

$$\Rightarrow\left[\begin{array}{cccccc}1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0\\ 0 & \frac{1}{4} & \frac{5}{4} \vdots & 1 & -\frac{1}{4} & 0\\ 3 & 5 & 3 & 0 & 0 & 1\end{array}\right] R_2 - R_1 \Rightarrow \left[\begin{array}{cccccc}1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0\\ 0 & \frac{1}{4} & \frac{5}{4} \vdots & 1 & -\frac{1}{4} & 0\\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1\end{array}\right] \quad R_3 - 3R_1$$

$$\Rightarrow\left[\begin{array}{cccccc}1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0\\ 0 & \frac{11}{4} & \frac{15}{4} \vdots & 0 & -\frac{3}{4} & 1\\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0\end{array}\right] \quad R_{23} \Rightarrow \left[\begin{array}{cccccc}1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0\\ 0 & 1 & \frac{15}{11} \vdots & 0 & -\frac{3}{11} & \frac{4}{11}\\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0\end{array}\right] \quad \frac{4}{11}R_2$$

$$\Rightarrow\left[\begin{array}{cccccc}1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0\\ 0 & 1 & \frac{15}{11} \vdots & 0 & -\frac{3}{11} & \frac{4}{11}\\ 0 & 0 & \frac{10}{11} & 1 & -\frac{2}{4} & -\frac{1}{11}\end{array}\right] \quad R_3 - \frac{1}{4}R_2 \Rightarrow \left[\begin{array}{cccccc}1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0\\ 0 & 1 & \frac{15}{11} \vdots & 0 & -\frac{3}{11} & \frac{4}{11}\\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{array}\right] \qquad \frac{11}{10}R_3$$

$$\Rightarrow\left[\begin{array}{cccccc}1 & \frac{3}{4} & 0 & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40}\\ 0 & 1 & 0 \vdots & -\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{array}\right] \quad R_1 + \frac{1}{4}R_3 \; and \; R_2 - \frac{15}{11}R_3$$

$$\Rightarrow\left[\begin{array}{cccccc}1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5}\\ 0 & 1 & 0 \vdots & -\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{array}\right] \quad R_1 - \frac{3}{4}R_2$$

Hence

$$A^{-1} = \left[\begin{array}{ccc}\frac{7}{5} & \frac{1}{5} & -\frac{2}{5}\\ -\frac{3}{2} & 0 & \frac{1}{2}\\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{array}\right]$$

QUESTION

Find inverse using Guass's Jordan Elimination Method $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$

ANSWER

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \quad R_2 - 4R_1 \; and \; R_3 - 3R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \quad -1R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \quad R_3 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \quad -\frac{1}{10}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{-1}{10} & \frac{1}{5} & \frac{1}{10} \\ 0 & 1 & 0 & \frac{-3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \quad R_1 - R_3 \; and \; R_2 - 5R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \quad R_1 - R_2$$

Hence

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

QUESTION: **Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$**

SOLUTION: **we first find the co-factor of the elements of A**

a_{11}= $(-1)^2(2+1)$= 3

a_{12}= $(-1)^3(-1)$= 1

a_{13}= $(-1)^4(0-2) =$ -2

a_{21}= $(-1)^3(0+2)$= -2

a_{22}=$(-1)^4(1-2)$= -1

a_{23}= $(-1)^5(-1-0) = 1$

a_{31}= $(-1)^4(0-4) = -4$

a_{32}= $(-1)^5(1-0) = -1$

a_{33}= $(-1)^6(2-0) = 2$

Thus $\left[A_{ij}\right]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$

$$adjA = \left[A'_{ij}\right]_{3\times3} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix} \qquad and \qquad |A| = -1$$

So $A^{-1} = \frac{1}{|A|}\ adjA = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$ after putting the values.

HESSENBERG MATRIX: **Matrix in which either the upper or lower triangle is zero except for the elements adjacent to the main diagonal.**

If the upper triangle has the zeroes, the matrix is the Lower Heisenberg and vice versa.

SPARSE: **A coefficient matrix is said to be sparse if many of the matrix entries are known to be zero.**

ORTHOGONAL MATRIX: **A "$n \times n$" matrix "M" is called orthogonal if**

$$MM^t = I \ \ i.e.\ A^t = A^{-1}$$

PERMUTATION MATRIX

A "$n \times n$" matrix $P = [P_{ij}]$ is a permutation matrix obtained by rearranging the rows of the identity matrix "I_n". This gives a matrix with precisely one non-zero entry in each row and in each column and each non-zero entry is "1"

For example $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

CONVERGENT MATRIX

We call a "$n \times n$" matrix "M" convergent if $\lim_{k\to\infty}(M^k)_{ij} = 0$ for each i, j=0, 1, 2 …n

Consider $M = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \implies M^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}$

Then $\lim_{k\to\infty}(\frac{1}{2})^k = 0$ **and** $\lim_{k\to\infty}\frac{k}{2^{k+1}} = 0 \implies M$ **is convergent.**

LOWER TRIANGULATION MATRIX

A matrix having only zeros above the diagonal is called Lower Triangular matrix.

(OR)

A "$n \times n$" matrix "L" is lower triangular if its entries satisfy $l_{ij} = 0 \ \ for\ i < j$

i.e. $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

UPPER TRIANGULATION MATRIX

A matrix having only zeros below the diagonal is called Upper Triangular matrix.

(OR)

A "$n \times n$" matrix "U" is upper triangular if its entries satisfy $u_{ij} = 0 \ for \ i > j$

i.e. $$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

CROUTS REDUCTION METHOD

In linear Algebra this method factorizes a matrix as the product of a Lower Triangular matrix and an Upper Triangular matrix.

Method also named as Cholesky's reduction method, triangulation method, or LU-decomposition (Factorization)

ALGORITHM

For a given system of equations $\sum_1^n x_i = m \, ; m \in Z$

1. **Construct the matrix "A"**
2. **Use "A=LU" (without pivoting) and "PA=LU" (with pivoting) where "P" is the pivoting matrix and find "u_{ij}, l_{ij}"**
3. **Use formula "AX=B" where "X" is the matrix of variables and "B" is the matrix of solution of equations.**
4. **Replace "AX=B" by "LUX=B" and then put "UX=Z" i.e. "LZ=B"**
5. **Find the values of "$Z_{i's}$" then use "Z=UX" find "$X_{i's}$" ; i=1, 2, 3, …….n**

ADVANTAGE/LIMITATION (FAILURE)

1. **Cholesky's method widely used in Numerical Solution of Partial Differential Equation.**
2. **Popular for Computer Programming.**
3. **This method fails if $a_{ii} = 0$ in that case the system is Singular.**

QUESTION

Solve the following system of equations using Crout's Reduction Method

$$5x_1 - 2x_2 + x_3 = 4$$

$$7x_1 + x_2 - 5x_3 = 8$$

$$3x_1 + 7x_2 + 4x_3 = 10$$

ANSWER

Let $$A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

One of the diagonals of L or U must be 1

Step I….

$$[A] = [L][U]$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

After multiplication on R.H.S

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$\Rightarrow l_{11} = 5, \quad l_{21} = 7, \quad l_{31} = 3$

$\Rightarrow l_{11}u_{12} = -2 \quad \Rightarrow 5u_{12} = -2 \quad \Rightarrow u_{12} = -2/5$

$\Rightarrow l_{11}u_{13} = 1 \quad \Rightarrow 5u_{13} = 1 \quad \Rightarrow u_{13} = 1/5$

$\Rightarrow l_{21}u_{12} + l_{22} = 1 \quad \Rightarrow 7(-2/5) + l_{22} = 1 \quad \Rightarrow l_{22} = 19/5$

$\Rightarrow l_{31}u_{12} + l_{32} = 7 \quad \Rightarrow 3(-2/5) + l_{32} = 7 \quad \Rightarrow l_{32} = 41/5$

$\Rightarrow l_{21}u_{13} + l_{22}u_{23} = -5 \quad \Rightarrow 7\left(\frac{1}{5}\right) + \left(\frac{19}{5}\right)u_{23} = -5 \Rightarrow u_{23} = -32/19$

$$\Rightarrow l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4 \quad \Rightarrow 3\left(\frac{1}{5}\right) + \left(\frac{41}{5}\right)\left(\frac{-32}{5}\right) + l_{33} = 4 \quad \Rightarrow l_{33} = 327/19$$

Step II…. **Put** $[A][X] = [B] \quad \Rightarrow \quad [L][U][X] = [B]$

Put [U] [X] = [Z] $[L][Z] = [B]$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & 19/5 & 0 \\ 3 & 41/5 & 327/19 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$$\Rightarrow 5z_1 = 4 \Rightarrow z_1 = 4/5 \Rightarrow 7z_1 + \frac{19}{5}z_2 = 8 \quad \Rightarrow 7\left(\frac{4}{5}\right) + \frac{19}{5}z_2 = 8 \Rightarrow z_2 = 12/19$$

$$\Rightarrow 3z_1 + \frac{41}{5}z_2 + \frac{327}{19}z_3 = 10 \qquad \Rightarrow \quad 3\left(\frac{4}{5}\right) + \frac{41}{5}\left(\frac{12}{19}\right) + \frac{327}{19}z_3 = 10 \quad \Rightarrow z_3 = 46/327$$

Step III…. **Since** **[U][X] = [Z]**

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & 1 & -32/19 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 12/19 \\ 46/327 \end{bmatrix}$$

$$\Rightarrow x_3 = 46/327$$

$$\Rightarrow x_2 - \frac{32}{19}x_3 = \frac{12}{19} \Rightarrow x_2 - \frac{32}{19}\left(\frac{46}{327}\right) = \frac{12}{19} \qquad \Rightarrow x_2 = 284/327$$

$$\Rightarrow x_1 - \left(\frac{2}{5}\right)x_2 + \frac{1}{5}x_3 = \frac{4}{5} \Rightarrow x_1 - \left(\frac{2}{5}\right)\left(\frac{284}{327}\right) + \frac{1}{5}\left(\frac{46}{327}\right) = \frac{4}{5}$$

$$\Rightarrow x_1 = 366/327$$

Hence required solutions are

$$\Rightarrow x_1 = 366/327, \qquad x_2 = 284/327, \qquad x_3 = 46/327$$

DIAGONALLY DOMINANT SYSTEM

Consider a square matrix "$A = \{a_{ij}\}$" then system is said to be Diagonally Dominant if
$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}| \qquad ; i = 1, 2, 3, \ldots \ldots \ldots \ldots . . n$

If we remove equality sign, then "A" is called strictly diagonally dominant and 'A' has the following properties

- **'A' is regular, invertible, its inverse exist and Ax = b has a unique solution.**
- **Ax = b can be solved by Gaussian Elimination without Pivoting.**

For example $\quad A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Then non- symmetric matrix 'A' is strictly diagonally dominant because

$|7| > |2| + |0| \quad ; \qquad |5| > |3| + |-1| \qquad ; \qquad |-6| = |6| > |0| + |5|$

But 'B' and 'A^t' are not strictly diagonally dominant (Check!)

NORM: **A norm measures the size of a matrix.**

Let "$x \in R^n$ or $x \in R^{n\times n}$" then $\|x\|$ satisfies

- $\|x\| \geq 0$
- **Iff x =0 then** $\|x\| = 0$
- $\|ax\| = |a| \, \|x\|$ **Where 'a' is constant.**
- $\|x + y\| \leq \|x\| + \|y\|$ **i.e. Triangular inequality**

INFINITY NORM $\|X\|_\infty$

The infinity (maximum) norm of a matrix 'X' is

$\|X\|_\infty = maximum\ of\ absolut\ values\ of\ components\ of\ "X" = \max_{1\leq i\leq n} |x_i|$

Consider $X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$

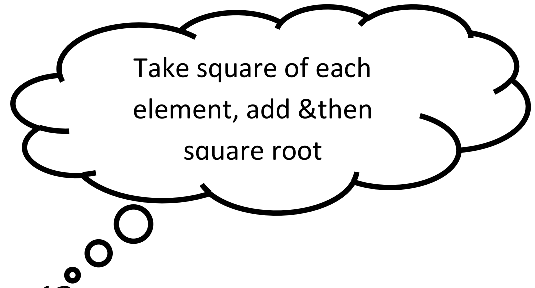
$\|X\|_\infty = maximum\ of\ absolute\ row\ sum = max \begin{bmatrix} 1+2+3 \\ 4+5+6 \\ 7+1+2 \end{bmatrix} = 15$

EUCLIDEAN NORM $\|X\|_2$

The Euclidean norm for the matrix 'X' is

$$\|X\|_2 = \left[\sum_{i=1}^{n} x_i^2\right]^{\frac{1}{2}}$$

We name it Euclidean norm because it represents the usual notation of distance from the origin in case x is in $R = R^1, R^2 \text{ or } R^3$



Consider

$$X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

$$\|X\|_2 = (1+4+9+16+25+36+49+1+4)^{1/2} = 12$$

USEFUL DEFINATIONS

Let "x_a" be an approximate solution of the linear system "Ax =b" then

The residual is the vector $r = b - Ax_a$

The backward error is the norm of residual $\|r = b - Ax_a\|$

The forward error is $\|x - x_a\|_\infty$

The relative backward error is $\frac{\|r\|_\infty}{\|b\|_\infty}$

The relative forward error is $\frac{\|x - x_a\|_\infty}{\|x\|_\infty}$

And error magnification factor is equals to $\frac{Relative\ forward\ error}{Relative\ backward\ error}$

CONDITION NUMBER

For a square matrix 'A' condition number is the maximum possible error magnification factor for solving Ax=b

Or The condition number of the "$n \times n$" matrix is defined as

$$cond\ (A) = \|A\|.\left\|A^{-1}\right\|$$

Remember: Identity matrix has the lowest condition number.

ILL CONDITION LINEAR SYSTEM

In practical application small change in the coefficient of the system of equations sometime gives the large change in the solutions of system. This type of system is called ill-condition linear system otherwise it is well-condition.

PROCEDURE (TEST, MEASURE OF CONDITION NUMBER)

- **Find determinant. If system is ill condition, then determinant will be very small.**
- **Find condition number.**
- **If condition number is very large then system of condition is ill-condition rather it is well-condition. Also determinant will be small.**

EXAMPLE: **Consider** $A = \begin{bmatrix} 2 & 1 \\ 2 & 0.1 \end{bmatrix} \Rightarrow |A| = 0.02 \; and \; \Rightarrow \|A\|_2 = 3.165$

$$\Rightarrow A^{-1} = \frac{adjA}{|A|} = \frac{1}{0.02}\begin{bmatrix} 1.01 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 50.5 & -50 \\ -100 & 100 \end{bmatrix}$$

$$and \; \left\|A^{-1}\right\|_2 = ((50.5)^2 + (-50)^2 + (-100)^2 + (100)^2)^{\frac{1}{2}} = 158.273$$

Now condition number = $\|A\|.\left\|A^{-1}\right\|$**= 500.93 (very large)**

Since condition Number is very large therefore system will be ill-condition.

Ill Conditioning Example

Here is a simple example of ill conditioning. Suppose that Ax = b is supposed to be

2x+6y= 8 ***and*** **2x+6.00001y= 8.00001**

The actual solution is x = 1, y = 1. Suppose further that due to representation error, the system on the machine is changed slightly to

2x+6y= 8 and 2x +5.99999y= 8.00002

The solution to this system is x = 10, y = −2, so you think the answer is (10,−2). When you check the answer by plugging these values into the actual system, you get

2(10) + 6(−2)=8 and 2(10) + 5.99999(−2)=7.99998

This seems to be acceptable, but of course (10,−2) is very far from the actual solution (1,1). This indicates that the system is badly ill conditioned.

Here are some things to consider if you have an ill conditioned system:

- **To identify if the matrix is ill conditioned, you can try 2 things. First, compute cond(A). This is relatively expensive and sometimes hard to interpret because the value may be in an intermediate range. Second, you can introduce deliberate "representation errors" by slightly perturbing one or more elements in A. Call the new matrix A^0, and solve $A^0x^0 = b$. If $x \approx x^0$, then there is probably no ill conditioning. The danger here is that you might be unlucky, and chose the wrong element to perturb. But if you try this several times with different elements and all the solutions are about the same, then you have confidence that the matrix is well conditioned.**

EXAMPLE

If the system really is ill conditioned, there is no simple fix. Consider using Singular Value Decomposition (SVD Ill-Conditioned Matrices

Consider systems $x + y = 2$ THEN $x + 1.001y = 2$ $x + 1.001y = 2.001$ The system on the left has solution $x = 2, y = 0$ while the one on the right has solution $x = 1, y = 1$. The coefficient matrix is called ill-conditioned because a small change in the constant coefficients results in a large change in the solution. A condition number, defined in more advanced courses, is used to measure the degree of ill-conditioning of a matrix (≈ 4004 for the above).

In the presence of rounding errors, ill-conditioned systems are inherently difficult to handle. When solving systems where round-off errors occur, one must avoid ill-conditioned systems whenever possible; this means that the usual row reduction algorithm must be modified. Consider the system: $.001x + y = 1$ AND $x + y = 2$ We see that the solution is $x = 1000/999 \approx 1$, $y = 998/999 \approx 1$ which does not change much if the coefficients are altered slightly (condition number ≈ 4).
The usual row reduction algorithm, however, gives an ill-conditioned system. Adding a multiple of the first to the second row gives the system on the left below, then dividing by −999 and rounding to 3 places on 998/999 = .99899 ≈ 1.00 gives the system on the right:

$$.001x + y = 1 \qquad .001x + y = 1$$
$$-999y = -998 \qquad y = 1.00$$

The solution for the last system is $x = 0, y = 1$ which is wildly inaccurate (and the condition number is ≈ 2002).

This problem can be avoided using partial pivoting. Instead of pivoting on the first non-zero element, pivot on the largest pivot (in absolute value) among those available in the column.

In the example above, pivot on the x, which will require a permute first:

$$x + y = 2 \qquad x + y = 2 \qquad x + y = 2$$
$$.001x + y = 1 \qquad .999y = .998 \qquad y = 1.00$$

where the third system is the one obtained after rounding. The solution is a fairly accurate $x = 1.00, y = 1.00$ (and the condition number is 4).

JACOBI'S METHOD

Method also known as iterative method, simultaneous displacement method. We want to solve 'Ax = b' where "$A \in R^{n\times n}$" and 'n' is very large, 'A' is Sparse (with a large percent of zero entries) as well as 'A' is structured (i.e. the product 'Ax' can be computed efficiently). For this purpose, we can easily use Jacoby's.

ALGORITHM

We want to solve Ax=b writes it out

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots \dots \dots .. + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots \dots \dots .. + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad \vdots \qquad + \cdots \dots \dots . \qquad = \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots \dots \dots .. + a_{nn}x_n = b_n \end{cases}$$

Rewrite it in another way

$$\begin{cases} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots \dots \dots \dots - a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots \dots \dots \dots - a_{2n}x_n) \\ \quad \vdots \qquad\qquad \vdots \qquad \dots \dots \dots . \qquad \vdots \\ x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots \dots \dots \dots - a_{n(n-1)}x_{n-1}) \end{cases}$$

Or in compact form

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij}x_j\right) \quad i = 1, 2, 3 \dots \dots \dots .. n$$

This gives the Jacoby's iteration.

Choose a start point (initial guess) i.e. $x^0 = (0, 0, 0)$

Apply $X^{k+1} = BX^k + C$ where $C_{ij} = \frac{b_i}{a_{ij}}$ and B can be defined as

$$B_{ij} = \begin{cases} -\frac{a_{ij}}{a_{ii}} & i \neq j \\ 0 & i = j \end{cases}$$

STOP CRITERIA

X^k close enough to X^{k-1} for example $\|X^k - X^{k-1}\| \leq \epsilon$ for certain vector norms.

Residual $r^k = Ax^k - b$ is small for example $\|r^k\| \leq \epsilon$

CONVERGENC CRITERIA

Sufficient condition for the convergence of Jacobi's is

$$\|X\|_2 < 1 \quad or \quad |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \qquad i = 1, 2, \ldots \ldots \ldots \ldots .. n$$

Jacobi method also called method of simultaneous displacement why?

Because no element of x_i^{k+1} is in this iteration until every element is computed.

KEEP IN MIND

- **Jacobi method is valid only when all "$a_{i's}$" are non-zeroes. (OR) the elements can rearrange for measuring the system according to condition. It is only possible if [A] is invertible i.e. inverse of 'A' exist.**
- **For fast convergence system should be diagonally dominant.**
- **Must make two vectors for the computation "X^k" and "X^{k+1}"**
- **System (method) is important for parallel computing.**

QUESTION: **Find the solution of the system of equation using Jacobi iterative method for the first five iterations.**

$$83x + 11y - 4z = 95 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (i)$$

$$3x + 52y + 13z = 104 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (ii)$$

$$3x + 8y + 29z = 71 \quad \ldots \ldots .. \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (iii)$$

ANSWER

$$(i) \Rightarrow \qquad x = \frac{95}{83} - \frac{11}{83}y + \frac{4}{83}z$$

$$(ii) \Rightarrow \qquad y = \frac{104}{52} - \frac{7}{52}x - \frac{13}{52}z$$

$$(iii) \Rightarrow \qquad z = \frac{71}{29} - \frac{8}{29}y - \frac{3}{29}x$$

Taking initial guess as (0, 0, 0) and using formula $X^{k+1} = BX^k + C$

Put k = 0 for first iteration

$$x^{(1)} = \frac{95}{83} - \frac{11}{83}(0) + \frac{4}{83}(0) = \frac{95}{83} = 1.1446$$

$$y^{(1)} = \frac{104}{52} - \frac{7}{52}(0) - \frac{13}{52}(0) = \frac{104}{52} = 2$$

$$z^{(1)} = \frac{71}{29} - \frac{8}{29}(0) - \frac{3}{29}(0) = \frac{71}{29} = 2.4483$$

$$\Rightarrow \qquad \left(x^{(1)}, \quad y^{(1)}, \quad z^{(1)}\right) = (1.1446, \quad 2, \quad 2.4483)$$

Put k = 1 for second iteration

$$x^{(2)} = \frac{95}{83} - \frac{11}{83}(2) + \frac{4}{83}(2.4483) = 0.9976$$

$$y^{(2)} = \frac{104}{52} - \frac{7}{52}(1.4466) - \frac{13}{52}(2.4483) = 1.2339$$

$$z^{(2)} = \frac{71}{29} - \frac{8}{29}(2) - \frac{3}{29}(1.1446) = \frac{71}{29} = 1.7781$$

$$\Rightarrow \qquad \left(x^{(2)}, \quad y^{(2)}, \quad z^{(2)}\right) = (0.9976, \quad 1.2339, \quad 1.7781)$$

Put k = 2 for third iteration

$$x^{(3)} = \frac{95}{83} - \frac{11}{83}(1.2339) + \frac{4}{83}(1.7781) = 1.0668$$

$$y^{(3)} = \frac{104}{52} - \frac{7}{52}(0.9976) - \frac{13}{52}(1.7781) = 1.4212$$

$$z^{(3)} = \frac{71}{29} - \frac{8}{29}(1.2339) - \frac{3}{29}(0.9976) = 2.0046$$

$$\Rightarrow \qquad \left(x^{(3)}, \quad y^{(3)}, \quad z^{(3)}\right) = (1.0668, \quad 1.4212, \quad 2.0046)$$

Put k = 3 for fourth iteration

$$x^{(4)} = \frac{95}{83} - \frac{11}{83}(1.4212) + \frac{4}{83}(2.0046) = 1.0529$$

$$y^{(4)} = \frac{104}{52} - \frac{7}{52}(1.0668) - \frac{13}{52}(2.0046) = 1.3553$$

$$z^{(4)} = \frac{71}{29} - \frac{8}{29}(1.4212) - \frac{3}{29}(1.0668) = \frac{71}{29} = 1.9451$$

$$\Rightarrow \qquad \left(x^{(4)}, \quad y^{(4)}, \quad z^{(4)}\right) = (1.0529, \quad 1.3553, \quad 1.9451)$$

Put k = 4 for fifth iteration

$$x^{(51)} = \frac{95}{83} - \frac{11}{83}(1.3551) + \frac{4}{83}(1.9451) = 1.0587$$

$$y^{(5)} = \frac{104}{52} - \frac{7}{52}(1.0529) - \frac{13}{52}(1.9451) = 1.3726$$

$$z^{(5)} = \frac{71}{29} - \frac{8}{29}(1.3553) - \frac{3}{29}(1.0529) = 1.9655$$

$$\Rightarrow \qquad \left(x^{(5)}, \quad y^{(5)}, \quad z^{(5)}\right) = (1.0587, \quad 1.3726, \quad 1.9655)$$

GUASS SEIDEL ITERATION METHOD

Guass's Seidel method is an improvement of Jacobi's method. This is also known as method of successive displacement.

ALGORITHM

In this method we can get the value of "x_1"from first equation and we get the value of "x_2" by using "x_1" in second equation and we get "x_3" by using "x_1" and "x_2" in third equation and so on.

ABOUT THE ALGORITHM

- **Need only one vector for both "x^k" and "x^{k+1}" save memory space.**
- **Not good for parallel computing.**
- **Converge a bit faster than Jacobi's.**

<u>How Jacobi method is accelerated to get Guass Seidel method for solving system of Linear Equations.</u>

In Jacobi method the (r+1)th approximation to the system $\sum_{j=1,j\neq i}^{n} a_{ij}x_j = b_i$ is given by $x_i^{r+1} = \frac{b_i}{a_{ii}} - \sum_{j=1,j\neq i}^{n} \frac{a_{ij}}{a_{ii}} x_j^r$; $r, j = 1, 2, 3, \dots \dots \dots \dots \dots n$ from which we can observe that no element of x_i^{r+1} replaces x_i^r entirely for next cycle of computations. However, this is done in Guass Seidel method. Hence called method of Successive displacement.

QUESTION: **Find the solutions of the following system of equations using Guass Seidel method and perform the first five iterations.**

$$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$

$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$

ANSWER

$$x_1 = 0.5 + 0.25x_2 + 0.25x_3$$

$$x_2 = 0.5 + 0.25x_1 + 0.25x_4$$

$$x_3 = 0.25 + 0.25x_1 + 0.25x_4$$

$$x_4 = 0.25 + 0.25x_2 + 0.25x_3$$

For first iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

For second iteration using $(0.5, 0.5, 0.25, 0.25)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0.25) + 0.25(0.25) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0.5) + 0.25(0.25) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0.5) + 0.25(0.25) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0.25) + 0.25(0.25) = 0.25$$

For third iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

For fourth iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

For fifth iteration using $(0, 0, 0, 0)$ we get

$$x_1^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25(0) + 0.25(0) = 0.5$$

$$x_3^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

$$x_4^{(1)} = 0.25 + 0.25(0) + 0.25(0) = 0.25$$

EIGENVALUE , EIGNVECTOR

Suppose ‘A’ is a square matrix. The number ‘λ' is called an Eignvalue of ‘A’ if there exist a non-zero vector ‘x’ such that

$$Ax = \lambda x \quad (or) \quad (A - \lambda I)x = 0$$

And corresponding non-zero solution vector ‘x’ is called an Eigenvector.

Largest Eigenvalue is known as Dominant Eigenvalue.

CHARACTERISTIC POLYNOMIAL

The polynomial defined by "$P(\lambda) = \det(A - \lambda I)$" is called characteristics polynomial.

SPECTRUM OF MATRIX

Set of all eignvalues of ‘A’ is called spectrum of ‘A’.

Write characteristic equation of

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$$

SPECTRAL RADIUS

The Spectral radius P(A) of a matrix ‘A’ is defined by

$$P(A) = max|\lambda| \qquad \text{Where ‘}\lambda'\text{ is an Eignvalue f ‘A’.}$$

SPECTRAL NORM

Let "λ_i" be the largest Eigenvalue of AA^* or A^*A where A^* is the conjugate transpose of “A” then the spectral norm of the matrix “A” is defined as

$$\sigma(A) = \sqrt{\lambda_i}$$

DETERMINANT OF A MATRIX

The determinant of “$n \times n$” matrix is the product of its Eigenvalues.

TRACE OF A MATRIX

The sum of diagonal elements of “$n \times n$” matrix is called the Trace of matrix “A”

This is also defined as the sum of Eigenvaluse of a matrix is Trace of it

THE POWER METHOD

The power method is an iterative technique used to determine the dominant eigenvalue of a matrix. i.e the eigenvalue with the largest magnitude.

Method also called RELEIGH POWER METHOD

ALGORITHIM

I. **Choose initial vector such that largest element is unity.**
II. **This normalized vector $V^{(0)}$ ia premultiplied by 'nxn' matrix $[A]$.**
III. **The resultant vector is again normalized.**
IV. **Continues this process untill required accuracy is obtained.**

At this point result looks like $\boldsymbol{U^{(k)} = [A]\, V^{(k-1)} = q_k\, V^{(k)}}$

Here 'q_k' is the desired largest Eigen value and '$v^{(k)}$' is the corresponding EigenVector.

CONVERGENCE

Power method Converges linearly , meaning that during convergence, the error decreases by a constant factor on each iteration step.

Question

How to find smallest Eigen value using power method?

Answer

Consider

$$[A]X = \lambda X$$

$$[A^{-1}][A]X = \lambda[A^{-1}]X$$

$$X = \lambda[A^{-1}]X$$

$$\Rightarrow [A^{-`1}]X = \frac{1}{\lambda}X \quad Required$$

Example

Find the Eigen value of largest modulus and the associated eigenvector of the matrix by power method

$$[A] = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}$$

Solution:

Let initial vector $V^{(0)}$ as $(0,0,1)^T$

You can take any other instead of $(0,0,0)$ which consist "0" and "1" like $(1,0,0)$ and $(1,1,1)$

(1). Using Formula $U^{(k)} = [A][V^{k-1}]$ **for K=1**

$$U^{(1)} = [A][V^{(0)}] = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} = 9 \begin{bmatrix} 9/9 \\ 5/9 \\ 1 \end{bmatrix} = 9 \begin{bmatrix} 0.222 \\ 0.556 \\ 1 \end{bmatrix} = q_1 V^{(1)}$$

(2). Using Formula $U^{(k)} = [A][V^{k-1}]$ **for K=2**

$$U^{(2)} = [A]V^{(1)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 0.222 \\ 0.556 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.112 \\ 7.556 \\ 10.778 \end{bmatrix} = 10.778 \begin{bmatrix} 0.382 \\ 0.701 \\ 1 \end{bmatrix} = q_2 V^{(2)}$$

(3). Using Formula $U^{(k)} = [A][V^{k-1}]$ **for K=3**

$$U^{(3)} = [A]V^{(2)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 0.382 \\ 0.701 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.867 \\ 8.631 \\ 11.548 \end{bmatrix} = 11.548 \begin{bmatrix} 0.421 \\ 0.747 \\ 1 \end{bmatrix} = q_3 V^{(3)}$$

(4). Using Formula $U^{(k)} = [A][V^{k-1}]$ **for K=4**

$$U^{(4)} = [A]V^{(3)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 0.421 \\ 0.747 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.083 \\ 8.925 \\ 11.757 \end{bmatrix} = 11.757 \begin{bmatrix} 0.432 \\ 0.759 \\ 1 \end{bmatrix} = q_4 V^{(4)}$$

(5). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=5

$$U^{(5)} = [A]V^{(4)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 0.432 \\ 0.759 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.141 \\ 9.005 \\ 11.814 \end{bmatrix} = 11.814 \begin{bmatrix} 0.435 \\ 0.762 \\ 1 \end{bmatrix} = q_5 V^{(5)}$$

(6). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=6

$$U^{(6)} = [A]V^{(5)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 0.435 \\ 0.762 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.156 \\ 9.026 \\ 11.829 \end{bmatrix} = 11.829 \begin{bmatrix} 0.436 \\ 0.763 \\ 1 \end{bmatrix} = q_6 V^{(6)}$$

(7). Using Formula $U^{(k)} = [A][V^{k-1}]$ for K=7

$$U^{(7)} = [A]V^{(6)} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 0.436 \\ 0.763 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.161 \\ 9.033 \\ 11.834 \end{bmatrix} = 11.834 \begin{bmatrix} 0.436 \\ 0.763 \\ 1 \end{bmatrix} = q_7 V^{(7)}$$

So largest Eigen value is $q = 11.834$ and corresponding Eigenvector is

$V = \begin{bmatrix} 0.436 \\ 0.763 \\ 1 \end{bmatrix}$ accurate to 3 decimals.

QUESTION: Find the smallest Eigen value of the matrix by power method.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$

SOLUTION

Put $A^{-1} = B \Rightarrow \quad A^{-1} = \frac{adj\ A}{|A|}$ ……………………… (1)

$a_{11} = (-1)^2(20+3) = +23$

$a_{12} = (-1)^3(20+6) = -26$

$a_{13} = (-1)^4(12-24) = -12$

$a_{21} = (-1)^3(-15-6) = 21$

$a_{22} = (-1)^4(5-12) = -7$

$a_{23} = (-1)^5(3+18) = -21$

$a_{31} = (-1)^4(-8+3) = -5$

$a_{32} = (-1)^5(-1-8) = 9$

$a_{33} = (-1)^6(4+12) = 16$

$$adjA = \begin{bmatrix} 23 & 21 & -5 \\ -26 & -7 & 9 \\ -12 & -21 & 16 \end{bmatrix}$$

$$and \qquad |A| = \begin{vmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{vmatrix} = 1(20+3) + 3(20+6) + 2(12-24) = 77$$

$$(1) \Rightarrow \qquad B = A^{-1} = \begin{bmatrix} 23/77 & 21/77 & -5/77 \\ -26/77 & -7/77 & 9/77 \\ -12/77 & -21/77 & 16/77 \end{bmatrix}$$

Now Taking Initial vector as $V^{(0)} = (0, 0, 1)^T$

$$U^{(1)} = [B]V^{(0)} = \begin{bmatrix} \frac{23}{77} & \frac{21}{77} & -\frac{5}{77} \\ -\frac{26}{77} & -\frac{7}{77} & \frac{9}{77} \\ -\frac{12}{77} & -\frac{21}{77} & \frac{16}{77} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.06 \\ 0.12 \\ 0.21 \end{bmatrix} = 0.21 \begin{bmatrix} -0.29 \\ 0.57 \\ 1 \end{bmatrix} = q_1 V^{(1)}$$

$$U^{(2)} = [B]V^{(1)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.21 \end{bmatrix} \begin{bmatrix} -0.29 \\ 0.57 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.17 \\ 0.10 \end{bmatrix} = 0.17 \begin{bmatrix} 0.06 \\ 1 \\ 0.59 \end{bmatrix} = q_2 V^2$$

$$U^{(3)} = [B]V^{(2)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & -0.21 \end{bmatrix} \begin{bmatrix} 0.06 \\ 1 \\ 0.59 \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.04 \\ -0.16 \end{bmatrix} = 0.25 \begin{bmatrix} 1 \\ 0.16 \\ 0.59 \end{bmatrix} = q_3 V^3$$

$$U^{(4)} = [B]V^{(3)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.12 \end{bmatrix} \begin{bmatrix} 1 \\ 0.16 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 0.30 \\ -0.28 \\ -0.07 \end{bmatrix} = 0.30 \begin{bmatrix} 1 \\ -0.93 \\ -0.23 \end{bmatrix} = q_4 V^{(4)}$$

$$U^{(5)} = [B]V^{(4)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.12 \end{bmatrix} \begin{bmatrix} 1 \\ -0.93 \\ -0.23 \end{bmatrix} = \begin{bmatrix} 0.06 \\ -0.28 \\ 0.04 \end{bmatrix} = 0.06 \begin{bmatrix} 1 \\ -4.67 \\ 0.67 \end{bmatrix} = q_5 V^{(5)}$$

$$U^{(6)} = [B]V^{(5)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.12 \end{bmatrix} \begin{bmatrix} 1 \\ -4.67 \\ 0.67 \end{bmatrix} = \begin{bmatrix} -1.00 \\ 0.16 \\ 1.24 \end{bmatrix} = 1.24 \begin{bmatrix} -0.81 \\ 0.13 \\ 1 \end{bmatrix} = q_6\, V^{(6)}$$

$$U^{(7)} = [B]V^{(6)} = \begin{bmatrix} 0.30 & 0.27 & -0.06 \\ -0.34 & -0.09 & 0.12 \\ -0.16 & -0.27 & 0.12 \end{bmatrix} \begin{bmatrix} -0.81 \\ 0.13 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.27 \\ 0.38 \\ 0.30 \end{bmatrix} = 0.38 \begin{bmatrix} -0.71 \\ 1 \\ 0.79 \end{bmatrix} = q_7\, V^{(7)}$$

Similarly check next repeated answer gives us Eigenvalue.

DIFFERENCE OPERATORS

DIFFERENCE EQUATION

Equation involving differences is called Difference Equation.

Solution of differential equation will be sequence of y_k values for which the equation is true for some set of consecutive integer 'k'.

Order of differential equation is the difference between the largest and smallest argument 'k' appearing in it.

DIFFERENCE OF A POLYNOMIAL

The "nth" difference of a polynomial of degree 'n' is constant, when the values of the independent variable are given at equal intervals.

FINITE DIFFERENCES.

Let we have a following linear D. Equation

$$y''(x) + p(x)y' + q(x)y = r(x) \quad ; a \le x \le b$$

Subject to the boundary conditions $\quad y(a) = \propto \quad and \quad y(b) = \beta$

Then the finite difference method consists of replacing every derivative in above Equation by finite difference approximations such as the central divided difference approximations

$$y'(x_i) \approx \frac{1}{2h}[y(x_i + 1) - y(x_i - 1)]$$

$$y''(x_i) \approx \frac{1}{h^2}[y(x_i + 1) - 2y(x_i) + y(x_i - 1)]$$

<u>Shooting Method</u> is a finite difference method.

FINITE DIFFERENCES OF DIFFERENT ORDERS

Supposing the argument equally spaced so that $x_{k+1} - x_k = h$ the difference of the $'y_k'$ values are denoted as

$\Delta y_k = y_{k-1} - y_k$ **And are called First differences.**

Second differences are as follows

$$\Delta^2 y_k = \Delta(\Delta y_k) = \Delta y_{k+1} - \Delta y_k = y_{k+2} - 2y_{k+1} + y_k$$

In General: $\quad \Delta^n y_k = \Delta^{n-1}\, y_{k+1} - \Delta^{n-1}\, y_k$ **And are called nth differences**

DIFFERENCE TABLE

The standard format for displaying finite differences is called difference table.

DIFFERENCE FORMULAS

Difference formulas for elementary functions somewhat parallel those of calculus. Example include the following

The differences of a constant function are zero. In symbol "$\Delta c = 0$" where 'c' denotes a constant.

For a constant time another function we have $\Delta(cu_k) = c\Delta u_k$

The difference of a sum of two functions is the sum of their differences

$$\Delta(u_k + v_k) = \Delta u_k + \Delta v_k$$

The 'linearity property' generalizes the two previous results.

$$\Delta(c_1 u_k + c_2 v_k) = c_1 \Delta u_k + c_2 \Delta v_k$$

Where c_1 and c_2 are constants.

PROVE THAT $\Delta(cy_k) = c\Delta y_k$

This is analogous to a result of calculus

$$\Delta(cy_k) = cy_{k+1} - cy_k = c(y_{k+1} - y_k) = c\Delta y_k$$

FOR A CONSTANT FUNCTION ALL DIFFERENCES ARE ZERO, PROVE!

Let $\forall k \quad ; c = y$ then for all 'k'

$\Delta y_k = y_{k+1} - y_k = c - c = 0$ **Where $y_k = c$ is a constant function.**

REMEMBER

The fundamental idea behind finite difference methods is the replace derivatives in the differential equation by discrete approximations, and evaluate on a grid to develop a system of equations.

COLLOCATION

Like the finite difference methods, the idea behind the collocation is to reduce the boundary value problem to a set of solvable algebraic equations.

However, instead of discretizing the differential equation by replacing derivative with finite differences, the solution is given a functional from whose parameters are fit by the method.

CRITERION OF APPROXIMATION

Some methods are as follows

i. **collocation** ii. **Osculation** iii. **Least square**

FORWARD DIFFERENCE OPERATOR 'Δ'

We define forward difference operator as $\Delta y_i = y_{i+1} - y_i \qquad i = 1, 2, \dots n-1$
Where y=f(x) (OR) $\Delta y_x = y_{x+h} - y_x$

<u>For first order</u>

Given function y=f(x) and a value of argument 'x' as x=a, a+h…… a+nh etc.
Where 'h' is the step size (increment) first order Forward Difference Operator is

$$\Delta f(a) = f(a+h) - f(a) \qquad OR \qquad \Delta y_i = y_{i+1} - y_i \quad \forall i = 1, 2, 3 \dots n-1$$

<u>For Second Order</u>

Let $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$
$= y_2 - 2y_1 + y_0$

For Third Order

$$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(y_2 - 2y_1 + y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0$$
$$= (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$\Rightarrow \Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ **are called leading differences**

<u>In General</u>: $\Delta^n y_0 = y_n - {}^n_1C y_{n-1} + {}^n_2C y_{n-2} + \cdots + (-1)^n y_n$

<u>Remark</u> ${}^n_rC = \frac{n!}{r!(n-r)!}$ *and* ${}^n_0C = {}^n_nC = 1$ *and* ${}^n_1C = {}_{n-1}^{\;\;n}C = n$

CONSTRUCTION OF FORWARD DIFFERENCE TABLE (Also called Diagonal difference table)

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\longrightarrow$ $\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\longrightarrow$ $\Delta^2 y_0$		
		$\longrightarrow$ $\Delta y_1 = y_2 - y_1$		$\longrightarrow$ $\Delta^3 y_0$	
x_2	y_2		$\longrightarrow$ $\Delta^2 y_1$		$\longrightarrow$ $\Delta^4 y_0$
		$\longrightarrow$ $\Delta y_2 = y_3 - y_2$		$\longrightarrow$ $\Delta^3 y_1$	
x_3	y_3		$\longrightarrow$ $\Delta^2 y_2$		
		$\longrightarrow$ $\Delta y_3 = y_4 - y_3$			
x_4	y_4				

QUESTION: Construct forward difference Table for the following value of 'X' and 'Y'

X	0.1	0.3	0.5	0.7	0.9	1.1	1.3
Y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

SOLUTION

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.1	0.003						
		→ 0.064					
0.3	0.067		→ 0.017				
		→ 0.081		→ 0.002			
0.5	0.148		→ 0.019		→ 0.001		
		→ 0.100		→ 0.003		→ 0	
0.7	0.248		→ 0.022		→ 0.001		→ 0
		→ 0.122		→ 0.004		→ 0	
0.9	0.370		→ 0.026		→ 0.001		
		→ 0.148		→ 0.005			
1.1	0.518		→ 0.031				
		→ 0.179					
1.3	0.697						

QUESTION

Express $\Delta^2 y_1$ and $\Delta^4 y_0$ in terms of the value of function y.

SOLUTION

$(I) \Rightarrow$ $\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_1) - (y_2 - y_1) = y_3 - 2y_2 + y_1$

$(II) \Rightarrow$ $\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$

$= \Delta^2 y_2 - \Delta^2 y_1 - (\Delta^2 y_1 - \Delta^2 y_0)$

$= \Delta y_3 - \Delta y_2 - (\Delta y_2 - \Delta y_1) - (\Delta y_2 - \Delta y_1) + (\Delta y_1 - \Delta y_0)$

$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$

QUESTION

Compute the missing values of y_n and Δy_n in the following table.

y_n	y_0	y_1	y_2	$y_3=6$	y_4	y_5	y_6
Δy_n	Δy_0	Δy_1	$\Delta y_2=5$	Δy_3	Δy_4	Δy_5	
$\Delta^2 y_n$	$\Delta^2 y_0=1$	$\Delta^2 y_1=4$	$\Delta^2 y_2=13$	$\Delta^2 y_3=18$	$\Delta^2 y_4=24$		

SOLUTION

$\Delta^2 y_0=1 \quad \Rightarrow \Delta y_1 - \Delta y_0 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$

$\Delta^2 y_1=4 \quad \Rightarrow \Delta y_2 - \Delta y_1 = 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$

$\Delta^2 y_2=13 \quad \Rightarrow \Delta y_3 - \Delta y_2 = 13 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$

$\Delta^2 y_3=18 \quad \Rightarrow \Delta y_4 - \Delta y_3 = 18 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$

$\Delta^2 y_4=24 \quad \Rightarrow \Delta y_5 - \Delta y_4 = 24 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$

$(2) \Rightarrow$ $\Delta y_2 - \Delta y_1=4$ **and** $\Delta y_2= 5 \Rightarrow 5-\Delta y_1=4 \Rightarrow \Delta y_1 = 1$

(1) $\Rightarrow$ $\Delta y_1 - \Delta y_0 = 1 \Rightarrow 1 - \Delta y_0=1 \Rightarrow \Delta y_0=0$

(3) $\Rightarrow$ $\Delta y_3 - \Delta y_2 = 13$ **And** $\Delta y_2 = 5 \Rightarrow \Delta y_3 - 5 = 13 \Rightarrow \Delta y_3=18$

(4) $\Rightarrow \Delta y_4 - \Delta y_3 = 18 \Rightarrow \Delta y_4 - 18 = 18 \Rightarrow \Delta y_4 = 36$

(5) $\Rightarrow \Delta y_5 - \Delta y_4 -= 18 \Rightarrow \Delta y_5 - 36 = 24 \Rightarrow \Delta y_5 = 60$

Now since we know that

$\Delta y_0 = y_1 - y_0 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(6)$

$\Delta y_1 = y_2 - y_1 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(7)$

$\Delta y_2 = y_3 - y_2 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(8)$

$\Delta y_3 = y_4 - y_3 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(9)$

$\Delta y_4 = y_5 - y_4 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(10)$

$\Delta y_5 = y_6 - y_5 \quad \dots\dots\dots\dots\dots\dots\dots\dots.(11)$

Since By table $y_3 = 6$ **and** $\Delta y_2 = 5$

$(8) \Rightarrow 5 = 6 - y_2 \Rightarrow y_2 = 1$

$(7) \Rightarrow \Delta y_1 = y_2 - y_1 \Rightarrow 1 = 1 - y_1 \Rightarrow y_1 = 0$

$(6) \Rightarrow \Delta y_0 = y_1 - y_0 \Rightarrow 0 = 0 - y_0 \Rightarrow y_0 = 0$

$(9) \Rightarrow \Delta y_3 = y_4 - y_3 \Rightarrow 18 = y_4 - 6 \Rightarrow y_4 = 24$

$(10) \Rightarrow \Delta y_4 = y_5 - y_4 \Rightarrow 36 = y_5 - 24 \Rightarrow y_5 = 60$

$(11) \Rightarrow \Delta y_5 = y_6 - y_5 \Rightarrow 60 = y_6 - 60 \quad \Rightarrow y_6 = 120$

QUESTION

Show that the value of 'y_n' can be expressed in terms of the leading value'y_0 'and the Binomial leading differences $\Delta y_0, \Delta^2 y_0 \dots\dots \Delta^n y_0$

SOLUTION

$$(1)\dots\dots\dots\dots\dots.. \begin{cases} \Delta y_0 = y_1 - y_0 \quad OR \quad y_1 = y_0 + \Delta y_0 \\ \Delta y_1 = y_2 - y_1 \quad OR \quad y_2 = y_1 + \Delta y_1 \\ \Delta y_2 = y_3 - y_2 \quad OR \quad y_3 = y_2 + \Delta y_2 \\ \quad and\ so\ on \dots\dots\dots\dots\dots \end{cases}$$

Similarly

$$(2)\dots\dots\dots\dots\dots.. \begin{cases} \Delta^2 y_0 = \Delta(\Delta y_0) = \Delta y_1 - \Delta y_0 \quad OR \quad \Delta y_1 = \Delta y_0 + \Delta^2 y_0 \\ \Delta^2 y_1 = \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 \quad OR \quad \Delta y_2 = \Delta y_1 + \Delta^2 y_1 \\ \quad and\ so\ on \dots\dots\dots\dots\dots \end{cases}$$

Similarly

$$(3)\dots\dots\dots\dots\dots.. \begin{cases} \Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 \quad OR \quad \Delta^2 \Delta y_1 = \Delta^2 y_0 + \Delta^3 y_0 \\ \Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1 \quad OR \quad \Delta^2 y_2 = \Delta^2 y_0 + \Delta^3 y_0 \\ \quad and\ so\ on \dots\dots\dots\dots\dots \end{cases}$$

Also from (2) and (3) we can write Δy_2 as

$$\Delta y_2 = (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0) = \Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0 \ldots\ldots\ldots\ldots (4)$$

From (1) and (4) we can write y_3 as

$$y_3 = y_2 + \Delta y_2 = (y_1 + \Delta y_1) + (\Delta y_1 + \Delta^2 y_1)$$

$$= (y_0 + \Delta y_0) + 2(\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0)$$

$$= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 = (1 + \Delta)^3 y_0$$

Similarly, we can symbolically write

$y_1 = (1 + \Delta) y_0$, $y_2 = (1 + \Delta)^2 y_0$, $y_3 = (1 + \Delta)^3 y_0$ **In general** $y_n = (1 + \Delta)^n y_0$

Hence $\quad y_n = y_0 + c_1^n \Delta y_0 + c_2^n \Delta^2 y_0 + ------ + c_n^n \Delta^n y_0 = \Sigma_{i=o}^n C_i^n \Delta^i y_0$

BACKWARD DIFFERENCE OPERATOR "∇"

We Define Backward Difference Operator as

$\nabla y_n = y_n - y_{n-1} \quad \forall\, n = 1, 2, \ldots\ldots i$ **(OR)** $\nabla f(x) = f(x) - f(x - h)$

(OR) $\quad \nabla y_x = y_x - y_{x-h}$

BACKWARD DIFFERENCE TABLE

X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
	$\longrightarrow$	$\nabla y_1 = y_1 - y_0$		
x_1	y_1		$\longrightarrow \nabla^2 y_2$	
	$\longrightarrow$	$\nabla y_2 = y_2 - y_1$		$\longrightarrow \nabla^3 y_3$
x_2	y_2		$\longrightarrow \nabla^2 y_3$	
	$\longrightarrow$	$\nabla y_3 = y_3 - y_2$		
x_3	y_3			

QUESTION

Show that any value of 'y' can be expressed in terms of $'y_n'$ and its backward differences.

SOLUTION

Since $y_{n-1} = y_n - \nabla y_n$ **And** $y_{n-2} = y_{n-1} - \nabla y_{n-1}$ ……………**(1)**

Also $\nabla y_{n-1} = \nabla y_n - \nabla^2 y_n$ ……………**(2)**

Thus $\nabla y_{n-1} = y_{n-1} - y_{n-2}$ **(Rearranging Above)**

(1) $\Longrightarrow y_{n-2} = y_{n-1} - \nabla y_n + \Delta^2 y_n = y_n - \nabla y_n - \nabla y_n + \nabla^2 y_n$

$$y_{n-2} = y_n - 2\nabla y_n + \Delta^2 y_n = (1 - 2\nabla + \nabla^2) y_n$$

Similarly We Can Show That $y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$

Symbolically above results can be written as

$y_{n-1} = (1 - \nabla) y_n , y_{n-2} = (1 - \nabla)^2 y_n$ ………

In General $y_{n-r} = (1 - \nabla)^r$

i.e. $y_{n-r} = y_n - {}^r_1C\, \nabla y_n + {}^r_2C\, \nabla^2 y_n - \ldots\ldots + (-1)^r \nabla^r y_n$

SHIFT OPERATOR "E"

Shift Operator defined as for y=f(x)

$E^n y_i = y_{i+n} \quad \forall\, i = 1, 2, \ldots\ldots,\ n = 1, 2, 3, \ldots\ldots$

OR $E^n f(x) = f(x + nh)$ **OR** $E^n y_x = y_{x+nh}$

"δ" CENTRAL DIFFERENT OPERATOR

Central Different Operator for y=f(x) defined as

$\delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}} \quad \forall\, i = 1, 2, \ldots . n$

(OR) $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$ **(OR)** $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$

TABLE

X	Y	δY	$\delta^2 Y$	$\delta^3 Y$
x_0	y_0			
		$\longrightarrow$ $\delta y_{\frac{1}{2}} = y_1 - y_0$		
x_1	y_1		$\longrightarrow$ $\delta^2 y_1$	
		$\longrightarrow$ $\delta y_{\frac{3}{2}}$		$\longrightarrow$ $\delta^3 y_{\frac{3}{2}}$
x_2	y_2		$\longrightarrow$ $\delta^2 y_2$	
		$\longrightarrow$ $\delta y_{\frac{5}{2}}$		
x_3	y_3			

AVERAGE OPERATOR "μ"

For y=f(x) Differential Operator defined as

$$\mu y_i = \frac{1}{2}\left[y_{i+\frac{1}{2}} + y_{i-\frac{1}{2}}\right] \qquad \forall\, i = 1, 2, \ldots \ldots \ldots n$$

(OR) $\mu f(x) = \frac{1}{2}[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)]$ **(OR)** $\mu y_x = \frac{1}{2}[y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}]$

DIFFERENTIAL OPERATOR "D"

For y=f(x) Differential Operator defined as $D^n f(x) = \frac{d^n}{dx^n}\ f(x) \quad \forall\, n$

SOME USEFUL RELATIONS

From the Definition of "Δ" and "E" we have

$$\Delta y_x = y_{x+h} - y_x = E y_x - y_x = (E-1) y_x \quad \Longrightarrow \quad \Delta = E - 1$$

Now by definitions of ∇ and E^{-1} we have

$$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1 - E^{-1}) y_x \quad \Longrightarrow \quad \nabla = 1 - E^{-1} = \frac{E-1}{E}$$

The definition of Operators 'δ'and 'E' gives

$$\delta y_x = y_{x+\frac{h}{2}} - y_{x-}\frac{h}{2} = E^{\frac{1}{2}} y_x - E^{\frac{-1}{2}} y_x = (E^{\frac{1}{2}} - E^{\frac{-1}{2}}) y_x \quad \Longrightarrow \delta = E^{\frac{1}{2}} - E^{\frac{-1}{2}}$$

The definition of 'μ' and 'E' Yields

$$\mu y_x = \frac{1}{2}\left[y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right] = \frac{1}{2}[E^{\frac{1}{2}} + E^{\frac{-1}{2}}]y_x \quad \Rightarrow \quad \mu = \frac{1}{2}(E^{\frac{1}{2}} + E^{\frac{-1}{2}})$$

Now Relation between 'D' and 'E' is as follows

Since $Ey_x = y_{x+h} = f(x + h)$

Using Taylor series expansion, we have

$$Ey_x = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \cdots \ldots\ldots\ldots\ldots$$

$$Ey_x = f(x) + h\,D\,f(x) + \frac{h^2}{2!} D^2 f(x) + \cdots \ldots$$

$$Ey_x = \left[1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \cdots \ldots\ldots\ldots\ldots\right] f(x)$$

$$E\,y_x = e^{hD}\,y_x \qquad \therefore e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \ldots\ldots\ldots$$

Taking 'Log' on both sides we get $Log\,E = hD$

<u>Hence, all the operators are expressed in terms of 'E'</u>

$PROVE\ THAT \quad E\nabla = \Delta = \delta E^{\frac{1}{2}}$

$$E\nabla = E(1 - E^{-1}) \qquad \therefore \nabla = 1 - E^{-1}$$

$$= E - EE^{-1} = E - 1 = \Delta$$

$$And \quad \delta E^{1/2} = (E^{\frac{1}{2}} - E^{-1/2})\,E^{1/2} \qquad \because \delta = E^{1/2} - E^{-1/2}$$

$$= E - 1 = \Delta$$

$PROVE\ THAT \quad \delta = 2\sin h\left(\frac{hD}{2}\right)$

$$Since \quad \delta = \left(E^{1/2} - E^{-1/2}\right) \qquad \because log\,E = hD \Rightarrow E = e^{hD}$$

$$= 2\left(\frac{E^{1/2} - E^{-1/2}}{2}\right) = 2\left(\frac{e^{hD/2} - e^{-hD/2}}{2}\right) = 2\,Sin\,h\left(\frac{hD}{2}\right)$$

$$PROVE\ THAT \quad \mu = 2\cosh\left(\frac{hD}{2}\right)$$

$$Since \quad \mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] = \frac{1}{2}\left[e^{\frac{hD}{2}} + e^{\frac{-hD}{2}}\right] = \cosh\left(\frac{hD}{2}\right)$$

$$Show\ that \quad \delta, \mu, E, \Delta, \nabla \quad Commute$$

$$\delta E f(x) = \delta f(x+h) = f(x+h+h) - f(x+h-h) = f(x+2h) - f(x)$$

$$E\delta f(x) = E[f(x+h) - f(x-h)] = E f(x+h) - Ef(x-h)$$

$$= f(x+h+h) - f(x-h+x) = f(x+2h) - f(x)$$

$$\Rightarrow \delta E f(x) = E\delta f(x) \qquad Commute$$

Now

$$\Delta\nabla(y_x) = \Delta(y_x - y_{x-h}) = \Delta y_x - \Delta y_{x-h}$$

$$= (y_{x+h} - y_x) - (y_{x-h+h} - y_{x-h}) = y_{x+h} - y_x - y_x - y_{x-h}$$

$$= y_{x+h} - 2y_x + y_{x-h}$$

$$And \quad \nabla\Delta(y_x) = \nabla(y_{x+h} - y_x) = \nabla y_{x+h} - \nabla y_x = (y_{x+h} - y_{x+h-h}) - (y_x - y_{x-h})$$

$$= y_{x+h} - y_x - y_x + y_{x-h} = y_{x+h} - 2y_x + y_{x-h} \Rightarrow \Delta\nabla(y_x) = \nabla\Delta(y_x) \ Commute$$

$$hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$$

$$Since \quad hD = \log E = \log(1+\Delta) \qquad \because E = 1+\Delta$$

$$= -\log E^{-1} = -\log(1-\nabla)$$

$$Also \quad \mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}(E - E^{-1})$$

$$= \frac{1}{2}\left(e^{hD} - e^{-hD}\right) \qquad \because E = e^{hD},\ E^{-1} = e^{-hD}$$

$$\mu\delta = \sinh(hD) \quad \Rightarrow hD = \sinh^{-1}(\mu\delta)$$

$PROVE\ THAT \quad 1+\delta^2\mu^2 = (1+\frac{\delta^2}{2})^2$

$Since \quad \mu\delta = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)\left(E^{1/2}-E^{-1/2}\right) = \frac{1}{2}(E-E^{-1})$

$\mu^2\delta^2 = \frac{1}{4}(E-E^{-1})^{\ 2} \qquad \because Squqring\ both\ sides$

$\mu^2\delta^2 = \frac{1}{4}(E^2+E^{-2}-2)$

$1+\mu^2\delta^2 = \frac{1}{4}(E^2+E^{-2}-2)+1 \qquad \because Adding\ '1'\ on\ both\ sides$

$1+\mu^2\delta^2 = \frac{E^2+E^{-2}-2+4}{4} = \frac{E^2+E^{-2}+2}{4} = \frac{(E+E^{-1})^2}{4}$

$1+\mu^2\delta^2 = \left(\frac{E+E^{-1}}{2}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$

$Also \quad \delta = (E^{\frac{1}{2}} - E^{\frac{-1}{2}})$

$\delta^2 = (E^{\frac{1}{2}} - E^{\frac{-1}{2}})^2 \quad \because Squaring\ Both\ sides.$

$\frac{\delta^2}{2} = \frac{1}{2}(E^{\frac{1}{2}} - E^{\frac{-1}{2}})^2 \quad \because deviding\ 2\ on\ Both\ sides$

$1+\frac{\delta^2}{2} = \frac{E+E^{-1}-2}{2}+1 \because Adding\ 1\ on\ Both\ sides$

$1+\frac{\delta^2}{2} = \frac{E+E^{-1}-2+2}{2} = \frac{1}{2}\ (\ E+E^{-1}\ ) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

$Combining\ (i)\ and\ (ii)\ we\ get\ the\ result.$

$PROVE\ THAT \qquad E^{\frac{1}{2}} = \mu + \frac{\delta}{2}$

$since \qquad \mu+\frac{\delta}{2} = (\frac{E^{\frac{1}{2}}+E^{\frac{-1}{2}}}{2}) + \frac{1}{2}(E^{\frac{1}{2}} - E^{\frac{-1}{2}}) = \frac{1}{2}[E^{\frac{1}{2}} + E^{\frac{-1}{2}} + E^{\frac{1}{2}} - E^{\frac{-1}{2}} = \frac{1}{2}\left(2E^{\frac{1}{2}}\right) = E^{\frac{1}{2}}$

PROVE THAT $\Delta = \frac{\delta^2}{2} + \delta . \sqrt{1+\frac{\delta^2}{4}}$

$since\ \delta = E^{\frac{1}{2}} - E^{\frac{-1}{2}} \Longrightarrow \delta^2 = (E^{\frac{1}{2}} - E^{\frac{-1}{2}})^2 \ \because Squaring$

$\frac{\delta^2}{2} = \frac{1}{2}(E^{\frac{1}{2}} - E^{\frac{-1}{2}})^2 \ \because deviding\ by\ (2) \ \ldots\ldots\ldots\ldots\ldots..(i)$

$Also\ \frac{\delta^2}{4} = \frac{1}{4}(E^{\frac{1}{2}} - E^{\frac{-1}{2}})^2 \Longrightarrow 1 + \frac{\delta^2}{4} = 1 + \frac{1}{4}(E^{\frac{1}{2}} - E^{\frac{-1}{2}})^2 \ \therefore adding\ one\ on\ side$

$\sqrt{1+\frac{\delta^2}{4}} = \sqrt{\left(\frac{4+E+E^{-1}-2}{4}\right)^2} \quad \because taking\ squre$ **root on both sides**

$\sqrt{1+\frac{\delta^2}{4}} = \sqrt{\left(\frac{E^{1/2}+E^{-1/2}}{4}\right)^2} \Longrightarrow \sqrt{1+\frac{\delta^2}{4}} = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{\frac{1}{2}}\right)$

$Now \quad \delta . \sqrt{1+\frac{\delta^2}{4}} = \delta \frac{\left(E^{\frac{1}{2}}+E^{\frac{-1}{2}}\right)}{2} = \frac{\left(E^{\frac{1}{2}}-E^{\frac{-1}{2}}\right)\left(E^{\frac{1}{2}}+E^{\frac{-1}{2}}\right)}{2} = \frac{E-E^{-1}}{2} \ldots\ldots\ldots\ldots\ldots.(\mathbf{ii})$

$Adding\ (i)\ and\ (ii)\ we\ get$

$$\frac{\delta^2}{2} + \delta . \sqrt{1+\frac{\delta^2}{4}} = \frac{(E^{\frac{1}{2}}-E^{\frac{-1}{2}})^2}{2} + \frac{(E-E^{-1})}{2} = \frac{E+E^{-1}-2+E-E^{-1}}{2} = \frac{2E-2}{2} = \frac{2(E-1)}{2}$$

$= E - 1 = \Delta \ \ forward\ difference\ operator$

PROVE THAT $\frac{\mu}{\sqrt{1+\frac{\delta^2}{4}}} = 1$

$$\Longrightarrow \quad \frac{\mu}{\sqrt{1+\frac{\delta^2}{4}}} = \mu\left[1+\frac{\delta^2}{4}\right]^{-\frac{1}{2}} = \mu\left[1+\frac{E^{\frac{1}{2}}-E^{-\frac{1}{2}}}{4}\right]^{-\frac{1}{2}} = \mu\left[\frac{4+\left(E^{\frac{1}{2}}\right)^2+\left(E^{-\frac{1}{2}}\right)^2-2}{4}\right]$$

$$= \mu\left[\left(\frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}\right)^2\right]^{-\frac{1}{2}} = \mu . \mu^{-1} = 1$$

$$\boldsymbol{PROVE\ THAT} \quad \mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$$

$$\mu\delta = \frac{1}{2}(E^{\frac{1}{2}} + E^{\frac{-1}{2}})(E^{\frac{1}{2}} - E^{\frac{-1}{2}}) = \frac{1}{2}(E - E^{-1})$$

$\textit{Now since}\ \ \Delta = E - 1 \quad \textit{therefore}\ E = 1 + \Delta$

$$\mu\delta = \frac{1}{2}[1 + \Delta - E^{-1}] = \frac{\Delta}{2} + \frac{1}{2}(1 - E^{-1}) = \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E-1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E}$$

$$= \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2} \qquad \because E - 1 = \Delta$$

$$\boldsymbol{PROVE\ THAT} \quad \mu\sigma = \frac{\Delta + \nabla}{2}$$

$$\mu\sigma = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{\frac{1}{2}}\right)\left(E^{\frac{1}{2}} - E^{\frac{1}{2}}\right) = \frac{1}{2}(E - E^{-1})$$

$\textit{Since}\ \Delta = E - 1\ \ \textit{and}\ \ \nabla = 1 - E^{-1} = \frac{E-1}{E}$

$\textit{therefore} \qquad \mu\sigma = \frac{1}{2}(1 + \Delta - 1 + \nabla) = \frac{\Delta+\nabla}{2}$

Show that operators "μ" and "E" commute

From the definition of "μ" and "E"

$$\mu E y_0 = \mu y_1 = \frac{1}{2}(y_{\frac{3}{2}} + y_{\frac{1}{2}})$$

$\textit{While} \qquad E\mu y_0 = \mu y_1 = \frac{1}{2}E\left(y_{\frac{1}{2}} + y_{\frac{-1}{2}}\right) = \frac{1}{2}\left(y_{\frac{3}{2}} + y_{\frac{1}{2}}\right) \Longrightarrow \mu E = E\mu$

$\Longrightarrow$ *"μ" and "E" commute*

INTERPOLATION

For a given table of values(x_k, y_k) $\forall$ $K = 0, 1, 2, \dots \dots n$. the process of estimating the values of "y=f(x)" for any intermediate values of "x = g(x)" is called "interpolation".

If g(x) is a Polynomial, Then the process is called "Polynomial" Interpolation.

ERROR OF APPROXIMATION

The deviation of g(x) from f(x) i.e. |f(x) – g (x)| is called Error of Approximation.

EXTRAPOLATION

The method of computing the values of 'y' for a given value of 'x' lying outside the table of values of 'x' is called Extrapolation.

REMARK

A function is said to interpolate a set of data points if it passes through those points.

INVERSE INTERPOLATION

Suppose $f \epsilon\, C\,[a, b]$, $f'(x) \neq o$ on [a, b] and f has non- zero 'p' in [a, b]
Let "$x_0, x_1 \dots \dots \dots . x_n$" be 'n+1' distinct numbers in [a, b] with $f\,(x_k) = y_k$ for each $k = 0, 1, 2 \dots . n$.
To approximate 'p' construct the interpolating polynomial of degree 'n' on the nodes "$y_0, y_1 \dots\dots\dots y_n$" for "$f^{-1}$"
Since "y_k=f (x_k)" and f (p) =0, it follows that f^{-1} (y_k) = X_k and p = f^{-1} (0).

"Using iterated interpolation to approximate $f^{-1}(0)$ is called iterated Inverse interpolation"

LINEAR INTERPOLATION FORMULA

$$f(x) = p_1(x) = f_0 + p(f_1 - f_0) = f_0 + p\Delta f_0$$

Where $x = x_0 + ph \quad \Rightarrow p = \frac{x - x_0}{h} \qquad 0 \leq P \leq 1$

QUADRATIC INTERPOLATION FORMULA

$$f(x) = p_2(x) = f_0 + p\Delta f_0 + \frac{P(P-1)}{2}\Delta^2 f_0$$

Where $x = x_0 + ph \quad \Rightarrow p = \frac{x - x_0}{h} \qquad 0 \leq P \leq 2$

ERRORS IN POLYNOMIAL INTERPOLATION

Given a function f(x) and $a \leq x \leq b$, a set of distinct points x_i $i = 1, 2, \dots \dots . n$ and $x_i \in [a, b]$
Let $P_n(x)$ be a polynomial of degree ≤ n that interpolates f(x) at x_i
i.e. $P_n(x_i) = f(x_i)$; $i = 1, 2, 3, \dots \dots \dots . . n$
Then Error define as "$\in (x) = f(x) - P_n(x)$"

REMARK

Sometime when a function is given as a data of some experiments in the form of tabular values corresponding to the values of independent variable 'X' then

1. **Either we interpolate the data and obtain the function "f(x)" as a polynomial in 'x' and then differentiate according to the usual calculus formulas.**
2. **Or we use Numerical Differentiation which is easier to perform in case of Tabular form of the data.**

DISADVANTAGES OF POLYNOMIAL INTERPOLATION

- **n-time differentiable**
- **big error in certain intervals (especially near the ends)**
- **No convergence result**
- **Heavy to compute for large "n"**

EXISTENCE AND UNIQUENESS THEOREM FOR POLYNOMIAL INTERPOLATION

Given $(x_i, y_i)_{i=0}^{n}$ with Xi's distinct there exists one and only one Polynomial $P_n(x)$ of degree $\leq n$ such that $P_n(x_i) = y_i$; $i = 1, 2, \dots \dots \dots . . n$

PROOF

Existence Ok from construction.
For Uniqueness:
Assume we have two polynomials P(x), q(x) of degree $\leq$ n both interpolate the data i.e.
$p(x_i) = y_i = q(x_i)$; $i = 1, 2, \dots \dots \dots . . n$
Now let $g(x) = P(x) - q(x)$ which will be a polynomial of degree $\leq$ n
Furthermore, we have $g(x_i) = p(x_i) - q(x_i) = y_i - y_i = 0$; $i = 0, 1, 2, \dots \dots \dots . . n$
So g(x) has 'n+1' Zeros. We must have g(x)$\equiv$ 0. Therefor p(x)$\equiv$ g(x).

REMEMBER: Using Newton's Forward difference interpolation formula we find the n-degree polynomial $'P_n'$ which approximate the function $f(x)$ in such a way that $'P_n'$ and ' f ' agrees at 'n+1' equally Spaced 'X' Values. So that

$$P_n(x_0) = f_0, P_n(x_1) \cdots \dots \dots \dots \dots \dots , P_n(x_n) = f_n$$

Where $f_0 = f(x_0)$, $f_1 = f(x_1) \dots \dots \dots \dots . \cdots f_n = f_{(x_n)}$ **Are the values of ' f ' in table.**

NEWTON FORWARD DIFFERENCE INTERPOLATION FORMULA

Newton's Forward Difference Interpolation formula is

$$f(x) = P_n(x)$$
$$= f(x_0) + P\Delta f(x_0) + \frac{P(P-1)}{2!}\Delta^2 f(x_0) + \cdots + \frac{P(P-1)\cdots(P-n+1)}{n!}\Delta^n f(x_0)$$

Where $x = x_0 + ph,\quad P = \frac{x-x_0}{h}\quad And\ 0 \le p \le n$

DERIVATION:

Let $y = f(x),\qquad x_0 = f(x_0)\quad And\quad x_n = x_0 + nh\quad \Rightarrow x = x_0 + ph$

$$f(x) = f(x_0 + ph) = E^p f(x_0) = (1+\Delta)^p f(x_0)\qquad \therefore E = 1 + \Delta$$

$$= [1 + P\Delta + \frac{P(P-1)}{2!} + \cdots + \frac{P(P-1)\cdots(P-n+1)}{n!}]f(x_0)$$

$$f(x) = f(x_0) + P\Delta f(x_0) + \cdots + \frac{P(P-1)\cdots P-n+1)}{n!}f(x_0)$$

CONDITION FOR THIS METHOD

- **Values of 'x' must have equal distance i.e. equally spaced.**
- **Value on which we find the function check either it is near to start or end.**
- **If near to start, then use forward method.**
- **If near to end, then use backward method.**

QUESTION

Evaluate $f(15)$ given the following table of values

X	**:**	**10**	**20**	**30**	**40**	**50**
f(x)	**:**	**46**	**66**	**81**	**93**	**101**

SOLUTION

Here '15' nearest to starting point we use Newtown's Forward Difference Interpolation.

X	Y	ΔY	Δ^2Y	Δ^3Y	Δ^4Y
10	46				
		20			
20	66		-5		
		15		2	
30	81		-3		-3
		12		-1	
40	93		-4		
		8			
50	101				

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\therefore x = x_0 + ph \Rightarrow 15 = 10 + P(10) \Rightarrow P = 0.5$$

$$f(15) = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2!}(-5) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(2)$$
$$+ \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(-3)$$

$$\Rightarrow f(15) = 56.8672$$

NEWTONS'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

Newton's Backward Difference Interpolation formula is

$$y_x = f(x) \approx P_n(x)$$

$$= f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots\cdots + \frac{P(P+1)(P+2)\cdots\cdots(P+n-1)}{n!}\nabla^n f(x_n)$$

$$Where\ x = x_n + ph,\ \ p = \frac{x - x_n}{h};\ \ -n \leq P \leq 0$$

DERIVATION: Let $y = f(x), x_n = f(x_n)\ and\ x = x_n + Ph$ Then

$$f(x_n + Ph) = E^P f(x_n) = (E^{-1})^{-P} f(x_n) \quad = (1 - \nabla)^{-P} f(x_n) \qquad \therefore E^{-1} = 1 - \nabla$$

Using binomial expansion $f(x) = \left[1 + P\nabla + \frac{P(P+1)}{2!}\nabla^2 + \frac{P(P+1)(P+2)}{3!}\nabla^3 + \cdots\right] f(x_n)$

$$f(x) = f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots\cdots$$

This is required Newton's Gregory Backward Difference Interpolation formula.

QUESTION: **For the following table of values estimate f(7.5)**

X	1	2	3	4	5	6	7	8
f(x)	1	8	27	64	125	216	343	512

SOLUTION

Since '7.5' is nearest to End of table, So We use Newton's Backward Interpolation.

X	Y	∇Y	$\nabla^2 Y$	$\nabla^3 Y$	$\nabla^4 Y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	243		42		
		169			
8	512				

Since $P = \frac{x - x_n}{h} \Rightarrow P = \frac{7.5-8}{1} \Rightarrow P = -0.5$

Now $y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$

$$y = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2!}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(6)$$

$$y = 512 - 84.5 - 5.26 - 0.375 = f(x) = 421.875$$

LAGRANGE'S INTERPOLATION FORMULA

For points $x_0, x_1 \dots\dots\dots\dots. x_n$ define the cardinal Function

$l_0, l_1 \dots\dots\dots\dots. l_n \in P^n$ (polynomial of n-degree)

$$l_i(x_j) = \begin{cases} 1 & i=j \\ o & i\neq j \end{cases} \qquad \mathrm{i = o, 1, 2, \dots..n}$$

The Lagrange form of interpolation Polynomial is $p_n(x) = \sum_{i=0}^{n} l_i(x) y_i$

DERIVATION OF FORMULA

Let y=f (x) be a function which takes the values $y_{0,} y_{1,} y_2 \dots\dots\dots y_n$ so we will obtain an n-degree polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + \dots\dots\dots\dots + a_n$

Now $(i) \dots\dots\dots\dots.$

$$\begin{cases} y = f(x) = a_0(x-x_1)(x-x_2)\cdots\cdots(x-x_n) \\ +a_1(x-x_0)(x-x_2)\cdots\cdots(x-x_n) \\ +a_2(x-x_0)(x-x_1)(x-x_3)\cdots\cdots(x-x_n) \\ \vdots \\ \vdots \\ +a_n(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) \end{cases}$$

Now we find the constants $a_0, a_1, \cdots\cdots a_n$

Put x= x_0 in (i)

$$(i) \Rightarrow \begin{cases} y = f(x) = a_0(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n) \\ +a_1(x_0-x_0)(x_0-x_2)\cdots\cdots(x_0-x_n) \\ +a_2(x_0-x_0)(x_0-x_1)(x_0-x_3)\cdots\cdots(x_0-x_n) \\ \vdots \\ \vdots \\ +a_n(x_0-x_0)(x_0-x_1)\cdots\cdots(x_0-x_{n-1}) \end{cases}$$

$$\Rightarrow \quad y_o = a_o(x_o-x_1)(x_o-x_2)\cdots\cdots(x_o-x_n)$$

$$\Rightarrow \quad a_0 = y_1 \div [(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_1)]$$

Now Put x=x_1 *in*

$$y_1 = f(x_1) = a_1(x_1-x_o)(x_1-x_2)\cdots\cdots(x_1-x_n)$$

$$\Rightarrow \quad a_1 = y_1 \div [(x_n-x_0)(x_n-x_2)\cdots\cdots(x_n-x_{n-1})]$$

Similarly

$$a_n = y_n \div [(x_n - x_0)(x_n - x_1)\cdots\cdots(x_n - x_{n-1})]$$

Putting all the values in (i) we get

$$y = f(x) = y_0 \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} + y_1 \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots + y_n \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}$$

$$\Rightarrow y = f(x) = l_0 y_0 + l_1 y_1 + l_2 y_2 + \cdots\cdots + l_n y_n = \sum_{k=0}^{n} l_k y_k$$

Where $l_k(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots\ldots(x_k-x_{k-1})(x-x_{k+1})\ldots\ldots(x_k-x_n)}$

ALTERNATIVELY DEFINE

$$\pi(x) = (x - x_0)(x - x_1)\ldots\ldots(x - x_n)$$

Then $\pi'(x) = (1 - 0)[(x - x_1)(x - x_2)\ldots\ldots(x - x_n)]$

$$+(1 - 0)[(x - x_0)(x - x_2)(x - x_3)\ldots\ldots(x - x_n)]\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(1 - 0)[(x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_{n-1})]$$

$$\pi'(x_k) = (x_k - x_0)(x_k - x_1)\ldots\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots\ldots(x_k - x_n)$$

$$l_k(x) = \frac{(x-x_k)}{(x-x_k)} \cdot \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots\ldots(x_k-x_{k-1})(x-x_{k+1})\ldots\ldots(x_k-x_n)}$$

Then $l_k(x) = \frac{\pi(x)}{(x-x_k)\pi'(x)}$

CONVERGENCE CRITERIA

Assume a triangular array of interpolation nodes $x_i = x_i^{(n)}$ exactly '$n+1$' distinct nodes for "$n = 0, 1, 2 \ldots\ldots\ldots i$"

$x_0^{(0)}$

$x_0^{(1)}$ $x_1^{(1)}$

$x_0^{(2)}$ $x_1^{(2)}$ $x_2^{(2)}$

$x_0^{(n)}$ $x_1^{(n)}$ $x_2^{(n)}\ldots\ldots\ldots x_n^{(n)}$

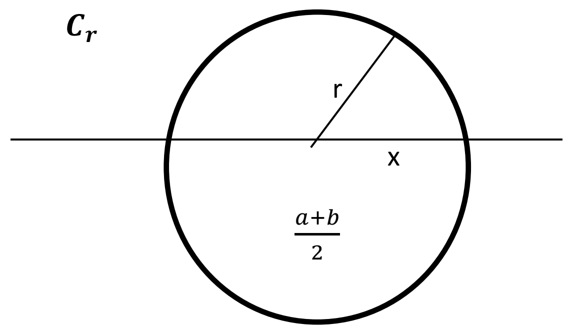
Further assume that all nodes $x_i^{(n)}$ are contained in finite interval $[a, b]$ then for each 'n' we define

$$P_n(x) = P_n\left(f;\ \ x_0^{(n)}, x_1^{(n)}, , , , , , , x_n^{(n)}\right), \quad x\epsilon[a, b]$$

Then we say method "converges" if $P_n\,(x) \to f(x)\ \ as\ \ n \to \infty$ uniformly for $x \in [a, b]$

(OR)

Lagrange's interpolation converges uniformly on [a, b] for on arbitrary triangular ret if nodes of 'f' is analytic in the circular disk 'C_r' centered at $\frac{a+b}{2}$ and having radius 'r' sufficiently large. So that $r > \frac{3}{2}(b - a)$ holds.



PROVE THAT $\int_a^b f(x)dx = \frac{b-a}{2}\,[f(a) + f(b)]$

PROOF: Using Lagrange's formula for $n = 1$ $\quad f(x) = \sum_{k=0}^{1} l_k\,(x) f(x_k) \quad k = 0, 1$

$$f(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

Integrating over $[a, b]$ when $x_o = a\,,\ \ x_1 = b$

$$\int_a^b f(x)\,dx = \int_a^b l_o(x) f(x_0)\,dx + \int_a^b l_1\,(x) f(x_1) dx$$

$$\int_a^b f(x)\,dx = f(x_0) \int_{x_0}^{x_1} l_0(x) dx + f(x_1) \int_{x_0}^{x_1} l_1(x) dx$$

Now $\quad l_0(x) = \frac{x-x_1}{x_0-x_1} \qquad l_1(x) = \frac{x-x_0}{x_1-x_0}$

$$\int_a^b f(x)\,dx = \ f(x_0) \int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1} dx + f(x_1) \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0} dx$$

Let $x = x_0 + ph \Rightarrow dx = hdp \quad as \quad x \to x_0 \quad then \quad p \to 0 \quad also \quad x \to x_1 \, then \quad p \to 1$

$$\int_a^b f(x)\,dx = \int_0^1 \frac{x_0+ph-x_1}{a-b} hdp. f(x_0) + f(x_1) \int_0^1 \frac{x_0+ph-x_0}{b-a} hdp$$
$$\int_a^b f(x)\,dx = \int_0^1 \frac{a+ph-b}{a-b} hdp. f(x_0) + f(x_1) \int_0^1 \frac{x_0+ph-x_0}{b-a} hdp$$

$$f(x) = f(x_0) \int_0^1 \frac{a-b+ph}{-h} hdp + f(x_1) \int_0^1 \frac{ph.hdp}{h} \qquad \therefore x_0 = a_1 \quad x_1 = b$$

$$f(x) = f(x_0) \int_0^1 \frac{-h+ph}{-1} dp + f(x_1) \int_0^1 ph\,dp = -hf(x_0) \int_0^1 (p-1)dp + f(x_1)\, h \int_0^1 p\,dp$$

$$f(x) = -hf(x_0) \left[\left| \frac{p^2}{2} \right|_0^1 - |p|_0^1 \right] + hf(x_1) \left| \frac{p^2}{2} \right|_0^1$$

$$f(x) = -hf(x_0) \left(\frac{1}{2} - 1 \right) + hf(x_1) \left(\frac{1}{2} \right) = -\frac{h}{2} f(x_0) + hf(x_0) + \frac{h}{2} f(x_1)$$

$$f(x) = \frac{h}{2} f(x_0) \; + \; \frac{h}{2} f(x_1) = \frac{h}{2} [f(x_0) + f(x_1)] = \frac{b-a}{2} [f(a) + f(b)]$$

Since $h = \frac{b-a}{n} = b - a \;\; for \; n = 1 \quad x_0 = a \quad and \;\; x_1 = b$

Hence the result

PROS AND CONS OF LAGRANGE'S POLYNOMIAL

- Elegant formula $(+)$
- Slow to compute, each $l_i(x)$ is different $(-)$
- Not flexible; if one change a point x_j , or add an additional point x_{n+1} one must re-compute all $l_{i's}(-)$

INVERSE LAGRANGIAN INTERPOLATION

Interchanging 'x' and 'y' in Lagrange's interpolation formula we obtain the inverse given by

$$x \approx l_n(y) = \sum_{i=0}^{n} \frac{l_k(y)}{l_k(y_k)} \; x_k$$

QUESTION

Find langrage's Interpolation polynomial fitting The points $y(1) = -3$,

$y(3) = 0, y(4) = 30, y(6) = 132,$ **Hence find $y(5) = ?$**

X: x_0**=1** x_1**=3** x_2**=4** x_3**=6**

Y: -3 0 30 132

ANSWER

Since $y(x) = l_o y_o + l_1 y_1 + l_2 y_2 + l_3 y_3$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_o + \frac{(x-x_o)(x-x_2)(x-x_3)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_o)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_o)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

By putting values, we get

$$y(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0) + \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

$$y(x) = \frac{1}{2}[-x^3 + 27x^2 - 92x + 60]$$

Put $x = 5$ **to get** $y(5)$

$$y(5) = \frac{1}{2}[-5^3 + 27(5^2) - 92(5) + 60] \implies \text{Y (5) =75}$$

DIVIDED DIFFRENCE

Assume that for a given value of $(x_1, y_1)(x_2, y_2) \dots .(x_n, y_n)$

$$y[x_0] = y(x_0) = y_0 \rightarrow y \text{ at } x_0$$

Then the first order divided Difference is defined as

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_o} \quad , y[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1} = a_1$$

The 2nd Order Difference is $y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_o, x_1]}{x_2, x_0} = a_2$

Similarly $y[x_0, x_1, x_2 \dots \dots x_n] = \frac{y[x_1, x_2, x_3 \dots \dots x_n] - y[x_o, x_1 \dots \dots x_{n-1}]}{x_n - x_o} = a_n$

DIVIDED DIFFERENCE IS SYMMETRIC

$$y[x_o,x_1] = \frac{y_1-y_0}{x_1-x_0} = \frac{-(y_o-y_1)}{-(x_0-x_1)} = y[x_1,x_0]$$

☺ **Also Newton Divided Difference is Symmetric**

NEWTON'S DIVIDED DIFFRENCE INTERPOLATION FORMULA

If $x_0,x_1 \ldots . x_n$ are arbitrarily Spaced (unequal spaced) Then the polynomial of degree 'n' through $(x_o,f_0)(x_1,f_1)\ldots\ldots.(x_n,f_n)$ where $f_j = f(x_j)$ is given by the newton's Devided difference Interpolation formula (Also known as Newton's General Interpolation formula) given by.

$$f(x) = (f_0) + (x-x_0)f[x_o,x_1] + (x-x_0)(x-x_1)f[x_o,x_1,x_2] + \cdots \ldots\ldots\ldots\ldots$$

$$\ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})f[x_o,x_1 \ldots . x_n]$$

DERIVATION OF FORMULA

Let $y = f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots \ldots\ldots\ldots..\ldots$

$\ldots\ldots\ldots\ldots\ldots \ldots . + a_n(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1}) \qquad \ldots\ldots\ldots\ldots..(i)$

$Put\ x = x_0 \Rightarrow y_1 = f(x_0) = a_0 + a_1(x_0-x_0) + 0 + 0 \ldots + 0 \Rightarrow a_0 = y_0 = y[x_o]$

Put $x = x_1 \quad \Rightarrow y_1 = f(x_1) = a_0 + a_1(x_1-x_0) + 0 + 0 \ldots + 0$

$$\Rightarrow a_1 = \frac{y_1-a_0}{x_1-x_0} = y[x_0,x_1] = \frac{y_1-y_o}{x_1-x_0} = \frac{y[x_1]-y[x_0]}{x_1-x_0} \qquad \because a_0 = y_0$$

Put $x = x_2 \quad y_2 = f(x_2) = a_0 + a_1(x_2-x_0) + a_2(x_2-x_0)(x_2-x_1) + 0 + 0 \ldots + 0$

$$\Rightarrow y_2 - a_0 - a_1(x_2-x_0) = a_2(x_2-x_0)(x_2-x_1)$$

$y_2 - y_0 - \left(\frac{y_1-y_0}{x_1-x_0}\right)(x_2-x_0) = a_2(x_2-x_0)(x_2-x_1) \quad \because$ **using above values**

$$\Rightarrow a_2 = \frac{y_2-y_0-(x_2-x_0)y[x_0,\ x_1]}{(x_2-x_0)(x_2-x_1)} = \frac{(y_2-y_1+y_1-y_0)-(x_2-x_0)y[x_0,x_1]}{(x_2-x_0)(x_2-x_1)}$$

$$a_2 = \frac{(y_2-y_1)+\frac{y_1-y_0}{x_1-x_0}(x_1-x_0)-(x_2-x_0)y[x_0,x_1]}{(x_2-x_0)(x_2-x_1)} = \frac{(y_2-y_1)+y[x_0\ ,\ x_1](x_1-x_0)-(x_2-x_0)y[x_0,x_1]}{(x_2-x_0)(x_2-x_1)}$$

$$a_2 = \frac{(y_2-y_1)+y[x_0\ ,\ x_1]\{x_1-x_0-x_2+x_0\}}{(x_2-x_0)(x_2-x_1)} = \frac{(y_2-y_1)+y[x_0\ ,\ x_1](x_1-x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$a_2 = \frac{\frac{y_2-y_1}{x_2-x_1}+\frac{y[x_0\ ,\ x_1](x_1-x_2)}{x_2-x_1}}{(x_2-x_0)} = \frac{y[x_1,x_2]-\frac{y[x_0\ ,\ x_1]}{(x_1-x_2)}(x_1-x_2)}{(x_2-x_0)} = \frac{y[x_1-x_2]-y[x_0\ ,x_1]}{(x_2-x_0)} = y[x_0, x_1, x_2]$$

Similarly $a_3 = y[x_0, x_1, x_2, x_3] \ldots \ldots \ldots \ldots \ldots \ldots a_n = y[x_0, x_1, x_2 \ldots a_n]$

$$(i) \Longrightarrow\ y = y[x_0] + (x - x_0)y[x_0, x_1] + .. + (x - x_0)(x - x_1)..(x - x_{n-1})y[x_o, x_1..x_n]$$

TABLE

X	Y	1[st] Order	2[nd] Order	3[nd] Order
x_0	y_0			
	$\rightarrow$	$y[x_o, x_1]$		
x_1	y_1	$\rightarrow$	$y[x_o, x_1, x_2]$	
	$\rightarrow$	$y[x_1, x_2]$	$\rightarrow$	$y[x_o, x_1, x_2, x_3]$
x_2	y_2	$\rightarrow$	$y[x_1, x_2, x_3]$	
	$\rightarrow$	$y[x_2, x_3]$		
x_3	y_3	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
x_n	y_n	⋮	⋮	⋮

EXAMPLE:

X	Y	$Y[x_o, x_1]$	$Y[x_o, x_1, x_2]$	$Y[x_o, x_1, x_2, x_3]$
2	25			
		$\frac{40-25}{5-2} = 5$		
5	40		$\frac{10-5}{7-2} = 1$	
		10		$\frac{0-1}{10-2} = -\frac{1}{8}$
7	60		$\frac{10-10}{10-5} = 0$	
		10		
10	90			

A RELATIONSHIP BETWEEN n[th] DIVIDED DIFFERENCE AND THE n[th] DARIVATIVE

Suppose "f" is n-time continuously differentiable and $x_0, x_1 \ldots x_n$ **are (n + 1) distinct numbers in [a, b] then there exist a number "§" in (a, b) such that** $f[x_0, x_1 \ldots x_n] = \frac{f^n(\S)}{n!}$

THEOREM

nth differences of a polynomial of degree 'n' are constant.

PROOF **Let us consider a polynomial of degree 'n' in the form**

$$y_x = a_o x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

Then $y_{x+h} = a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots + a_{n-1}(x+n) + a_n$

We now examine the difference of polynomial $\Delta y_x = y_{x+h} - y_x$

$$\Delta y_x = a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \cdots \dots \dots + a_{n-1}[x+h-x]$$

Binomial expansion yields

$$\Delta y_x = a_0(x^n + {}^nC_1\, x^{n-1}\, h + {}^nC_2\, x^{n-2}\, h^2 + \cdots + h^n - x^n)$$

$$+a_1(x^{n-1} + {}^{n-1}C_1\, x^{n-2}\, h + {}^{n-2}C_2\, x^{n-3}\, h^2 + \cdots + h^{n-1} - x^n) + \cdots + a_{n-1}\, h$$

$$\Delta y_x = a_0 nhx^{n-1} + \left[a_0\, {}^n_2Ch^2 + a_1\, {}^{n-1}_{\ \ 1}Ch\right]x^{n-2} + \cdots \dots \dots \dots .. + a_{n-1}\, h$$

Therefore $\Delta y_x = a_0 nhx^{n-1} + b' x^{n-2} + c' x^{n-3} + \cdots + k' x + l'$

Where b', c', k', l' **are constants involving 'h' but not 'x'**

Thus the first difference of a polynomial of degree 'n' is another polynomial of degree$(n-1)$

Similarly $\Delta^2 y_x = \Delta(\Delta y_x) = \Delta y_{x+h} - \Delta y_x$

$$= a_0\, nh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - \ x^{n-2}] + \cdots \dots . + k'(x+h-x)$$

$$\Delta^2 y_x = a_0\, n(n-1)h^2 x^{n-2} + b''\, x^{n-2} + C''\, x^{n-4} + \cdots + q''$$

Therefore $\Delta^2 y_x$ **is a polynomial of degree** $(n-2)$ **in 'x'**

Similarly, we can find the higher order differences and every time we observe that the degree of polynomial is reduced by one.

After differencing n-time we get

$$\Delta^n y_x = a_0(n-1)(n-2) \dots (2)(1)h^n = a_0(n!)h^n = \ constant.$$

This constant is independent of 'x' since $\Delta^n y_x$ **is constant ,** $\Delta^{n+1}\, y_x = 0$

Hence The $(n+1)th$ **and higher order differences of a polynomial of degree 'n' are zero.**

NEWTION'S DIVIDED DIFERENCE FORMULA WITH ERROR TERM

$$\mathbf{y}(\mathbf{x}) = y_0 + (x - x_0)y[x, x_0] \quad \dots\dots\dots\dots..(i)$$

$$\mathbf{y}[x, x_0] = y[x_o, x_1] + (x - x_1)y[x, x_o, x_1] \quad \dots\dots\dots\dots..(ii)$$

$$y[x, x_o, x_1] = y[x_o, x_1, x_2] + (x - x_2)y[x, x_o, x_1, x_2] \quad \dots\dots\dots\dots..(iii)$$

$$y[x, x_o, x_1, x_2] = y[x, x_o, x_1, x_2, x_3] + (x - x_3)y[x, x_o, x_1, x_2, x_3] \dots\dots\dots\dots..(iv)$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$y[x, x_o, x_1 \dots x_{n-1}] = y[x_0, x_1 \dots x_n] + (x, x_n)y[x, x_o, x_1 \dots x_n] \quad \dots\dots\dots\dots..(n)$$

Multiplying (ii) by $(x - x_0)$ **(iii)** $(x - x_0)\ (x - x_1) \dots\dots (n)$ **By** $(x - x_0)(x - x_1) \dots\dots (x_{n-1})$

And adding all Equation's

$$y(x) = y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1]$$

$$+ \cdots \dots\dots\dots\dots + (x - x_0)(x - x_1) \dots (x - x_n)y[x_0, x_1, x_2 \dots x_n]$$

Also last term will be $\in (x)$

LIMITATIONS OF NEWTON'S INTERPOLATION.

This formula used only when the values of independent variable 'x' are equally spaced. Also the differences of 'y' must ultimately become small. Its accuracy same as Lagrange's Formula but has the advantage of being computationally economical in the sense that it involves less numbers of Arithmetic Operations.

ERROR TERM IN INTERPOLATION

As we know that

$$y(x) = y_0 + (x - x_0)y[x_0, x_1] + \cdots \dots \dots .. \dots + (x - x_0)(x - x_1)..(x - x_{n-1})\, y[x_0, x_1 \dots x_n]$$

Approximated by polynomial $P_n(x)$ of degree 'n' the error term is

$$\in (x) = y(x) - P_n(x) \qquad \dots \dots \dots \dots \dots \dots \dots \dots .. (i)$$

$$\in (x) = (x - x_0)(x - x_1) \dots \dots \dots \dots .. (x - x_n)\, y[x, x_0, x_1 \dots x_n]$$

Let $\in (x) = \pi\,(x)\, y[x, x_0, x_1 \dots x_n] = k\,\pi\,(x) \qquad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots .. (ii)$

And $F(x) = y(x) - P_n(x) - k\,\pi(x)$

$F(x)$ **Vanish for** $x_0, x_1 \dots x_n$ **Choose arbitrarily $\overline{x}$ from them.**

Consider an interval 'I' which span the points $\overline{x}, x_0, x_1 \dots x_n$. Total number of points $(n+2)$ Then $F(x)$ vanish $(n+2)$ time by *Roll's theorem*

$F'(x)$ Vanish $(n+1)$ time, $F''(x)$ vanish n-time. Hence $F^{n+1}(x)$ vanish 1-time choose arbitrarily $x = \S$

$$\Rightarrow F^{n+1}(\S) = y^{n+1}(\S) - P_n^{n+1}(\S) - k\frac{d^{n+1}}{dx^{n+1}}\pi(\S)$$

$$\Rightarrow 0 = y^{n+1}(\S) - 0 - k\pi^{n+1}(\S) \qquad \therefore y^{n+1}(\S) = 0 \;\; and \;\; P^{n+1}(\S) = 0$$

$$\Rightarrow y^{n+1}(\S) = k\pi^{n+1}(\S) \qquad \Rightarrow k = \frac{y^{n+1}(\S)}{\pi^{n+1}(\S)}$$

$$if \qquad \pi^{n+1}(x) = (n+1)! \qquad \Rightarrow k = \frac{y^{n+1}(\S)}{(n+1)!} \qquad \Rightarrow k = y[x_0, x_1 \dots x_n]$$

$$(ii) \Rightarrow \qquad \in (x) = \frac{y^{n+1}(\S)}{(n+1)!}\,\pi(x)$$

NEWTON'S DIVIDED DIFFERENCE AND LAGRANGE'S INTERPOLATION FORMULA ARE IDENTICAL, PROVE!

Consider y = f(x) is given at the sample points x_0, x_1, x_2

Since by Newton's divided difference interpolation for x_0, x_1, x_2 **is given as**

$$y = y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2]$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + (x - x_0)(x - x_1)\frac{y[x_1,x_2] - y[x_0,x_1]}{x_2 - x_0}$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{(x-x_0)(x-x_1)}{x_2 - x_0}\right)\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{(x-x_0)(x-x_1)}{x_2 - x_0}\right)\left(\frac{y_2}{x_2 - x_1} + y_1\left\{\frac{-1 - 1}{(x_2 - x_1)(x_1 - x_0)}\right\} + \frac{y_0}{x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{(x-x_0)(x-x_1)}{x_2 - x_0}\right)\left(\frac{y_2}{x_2 - x_1} + y_1\left\{\frac{-(x_1 - x_0) - (x_2 - x_1)}{(x_2 - x_1)(x_1 - x_0)}\right\} + \frac{y_0}{x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{(x-x_0)(x-x_1)}{x_2 - x_0}\right)\left(\frac{y_2}{x_2 - x_1} + y_1\left\{\frac{x_0 - x_2}{(x_2 - x_1)(x_1 - x_0)}\right\} + \frac{y_0}{x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{y_2(x-x_0)(x-x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{y_1(x-x_0)(x-x_1)(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)} + \frac{y_0(x-x_0)(x-x_1)}{(x_2 - x_0)x_1 - x_0}\right)$$

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \left(\frac{y_2(x-x_0)(x-x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{y_1(x-x_0)(x-x_1)}{(x_2 - x_1)(x_1 - x_0)} + \frac{y_0(x-x_0)(x-x_1)}{(x_2 - x_0)x_1 - x_0}\right)$$

$$y = y_0\left[1 - \left(\frac{x - x_0}{x_1 - x_0}\right) + \frac{(x-x_0)(x-x_1)}{(x_2 - x_0)(x_1 - x_0)}\right] + y_1\left[\left(\frac{x - x_0}{x_1 - x_0}\right) - \frac{(x-x_0)(x-x_1)}{(x_2 - x_1)(x_1 - x_0)}\right] + y_2\left[\frac{(x-x_0)(x-x_1)}{(x_2 - x_0)(x_2 - x_1)}\right]$$

$$y = y_0\left[\frac{(x_1 - x_0)(x_2 - x_0) - (x - x_0) + (x-x_0)(x-x_1)}{(x_2 - x_0)(x_1 - x_0)}\right] + y_1\left[\frac{(x-x_0)(x_2 - x_1) - (x-x_0)(x-x_1)}{(x_2 - x_1)(x_1 - x_0)}\right] + y_2\left[\frac{(x-x_0)(x-x_1)}{(x_2 - x_0)(x_2 - x_1)}\right]$$

$$y = y_0\left[\frac{(x-x_1)(x-x_2)}{(x_0 - x_1)(x_0 - x_2)}\right] + y_1\left[\frac{(x-x_0)(x-x_2)}{(x_1 - x_0)(x_1 - x_2)}\right] + y_2\left[\frac{(x-x_0)(x-x_1)}{(x_2 - x_0)(x_2 - x_1)}\right]$$

This is Lagrange's form of interpolation polynomial.

Hence both Divided Difference and Lagrange's are identical.

SPLINE

A function 'S' is called a spline of degree 'k' if it satisfied the following conditions.

(i) **S is defined in the interval** $[a, b]$

(ii) S^r **is continuous on** $[a, b]$ $;0 \le r \le k-1$

(iii) **S is polynomial of degree** $less\ than\ equals\ to\ 'k'$ **on each subinterval** $[x_i, x_{i+1}]\ ; i = 1, 2, \dots, n-1$

CUBIC SPLINE INTERPOLATION

A function $S(x)$ **denoted by** $S_j(x)$ **over the interval** $\left[x_j, x_{j+1}\right]\ ; j = 0, 1, 2, \dots, n-1$

Is called a cubic spline interpolant if following conditions hold.

- $S_j(x_j) = f_j \qquad ; j = 0, 1, 2, \dots \dots \dots . . n$
- $S_{j+1}(x_{j+1}) = f_{j+1} \qquad ; j = 0, 1, 2, \dots \dots \dots . . n-2$
- $S_{j+1}{}'(x_{j+1}) = f'_{j+1} \qquad ; j = 0, 1, 2, \dots \dots \dots . . n-2$
- $S''_{j+1}(x_{j+1}) = f''_{j+1} \qquad ; j = 0, 1, 2, \dots \dots \dots . . n-2$

♠ **A spline of degree "3" is cubic spline.**

NATURAL SPLINE

A cubic spline satisfying these two additional conditions

$S''_1(x_1) = 0 \qquad and \qquad S''_{n-1}(x_n) = 0$

HERMIT INTERPOLATION

In Hermit interpolation we use the expansion involving not only the function values but also its first derivative.

Hermit Interpolation formula is given as follows

$$P(x) = \sum_{i=0}^{n}[1 - 2L_i'(x_i)(x - x_i)]\,[L_i(x_i)]^2 y_i + (x - x_i)[L_i(x_i)]^2 y_i'$$

EXAMPLE

Estimate the value of $y(1.05)$ using hermit interpolation formula from the following data

X	Y	Y'
1.00	1.00000	0.5000
1.10	1.04881	0.47673

Solution:

At first we compute $l_o(x) = \frac{x - x_1}{x_0 - x_1} = \frac{1.05 - 1.10}{1.00 - 1.10} = 0.5$

$$l_0'(x) = \frac{1}{x_0 - x_1} = \frac{1}{1.00 - 1.10} = -\frac{1}{0.10}$$

And $l_1\ (x) = \frac{x - x_0}{x_1 - x_0} = \frac{1.05 - 1.00}{1.10 - 1.00} = 0.5$

$$l_1'(x) = \frac{1}{x_1 - x_0} = \frac{1}{1.10 - 1.00} = \frac{1}{0.1}$$

Now putting the values in Hermit Formula

$$P(x) = \sum_{i=0}^{n}[1 - 2L_i'(x_i)(x - x_i)]\,[L_i(x_i)]^2 y_i + (x - x_i)[L_i(x_i)]^2 y_i'$$

We find

$$y(1.05) =$$
$$\left[1 - 2\left(-\frac{1}{0.1}\right)(0.05)\right]\left(\frac{1}{2}\right)^2 (1) + (0.05)\left(\frac{1}{2}\right)^2 (0.5) +$$
$$\left[1 - 2\left(\frac{1}{0.1}\right)(-0.05)\right]\left(\frac{1}{2}\right)^2 (1.04881) + (-0.05)\left(\frac{1}{2}\right)^2 (0.47673)$$
$$y(1.05) = 1.0247 \qquad \textit{required answer}$$

NUMARICAL DIFFERENTIATION

The problem of numerical differentiation is the determination of approximate values the derivatives of a function 'f' at a given point.

DIFFERENTIATION USING DIFFERENCE OPERATORS

We assume that the function $y = f(x)$ is given for the equally spaced 'x' values $x_n = x_0 + nh$ for $n = 0, 1, 2, \dots\dots\dots..$ to find the darivatives of such a tabular function, we proceed as follows;

USING FORWARD DIFFERENCE OPERATOR $'\Delta'$

Since $hD = \log E = \log(1 + \Delta) \qquad \therefore E = (1 + \Delta)$

$\Rightarrow D = \frac{1}{h}[\log(1 + \Delta)]$ **Where D is differential operator.**

$$\Rightarrow D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] \quad \dots\dots\dots\dots\dots.(i) \quad using\ Maclaurin\ series$$

Therefore

$$D\,f(x_0) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] f(x_0) = f'(x_0)$$

$$D\,f(x_0) = f'(x_0) = \frac{1}{h}\left[\Delta f(x_0) - \frac{\Delta^2}{2} f(x_0) + \frac{\Delta^3}{3} f(x_0) - \frac{\Delta^4}{4} f(x_0) + \cdots\dots\dots\dots\dots.\right]$$

$$Dy_0 = y_0' = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2}{2} y_0 + \frac{\Delta^3}{3} y_0 - \frac{\Delta^4}{4} y_0 \dots\dots\dots..\right]$$

Similarly, for second derivative

$$(i) \Longrightarrow \qquad D^2 = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} \dots\dots\dots..\right]^2$$

$$D^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots\dots\dots..\right] \quad \textbf{After solving}$$

$$D^2 y_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\dots\dots..\right] = y_0''$$

USING BACKWARD DIFFERENCE OPERATOR "∇"

Since $hD = \log E = \log(E^{-1})^{-1} = -1 log\, E^{-1} = -1\log(1-\nabla)$

Since $log(1-\nabla) = -\nabla - \frac{\nabla^2}{2} - \frac{\nabla^3}{2} - \cdots \ldots\ldots\ldots\ldots\ldots$ **therefore**

$$\Rightarrow D = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \cdots\right] \quad ..\ldots\ldots\ldots\ldots\ldots.(i)$$

Now $D\, f(x_n) = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \cdots\right] f(x_n) = f'(x_n)$

$$D\, f(x_n) = f'(x_n) = \frac{1}{h}\left[\nabla f(x_n) + \frac{\nabla^2}{2} f(x_n) + \frac{\nabla^3}{3} f(x_n) - \frac{\nabla^4}{4} f(x_n) + \cdots\right]$$

$$Dy_n = y'_n = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2}{2} y_n + \frac{\nabla^3}{3} y_n - \frac{\nabla^4}{4} y_n + \cdots\right]$$

Similarly, for second derivative squaring (i) we get

$$(i) \Longrightarrow \qquad D^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \cdots \ldots\ldots\ldots\ldots\right]$$

$$D^2 y_n = y''_n = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots \ldots\ldots.\ldots\right]$$

TO COMPUTE DARIVATIVE OF A TABULAR FUNCTION AT POINT NOT FOUND IN THE TABLE

Since

$$y(x_n + ph) =$$
$$f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots\cdots + \frac{P(P+1)(P+2)\cdots\cdots(P+n-1)}{n!}\nabla^n f(x_n) \quad \ldots\ldots\ldots\ldots(i)$$

$$Where \quad x = x_n + ph \qquad \Longrightarrow \; p = \frac{x-x_n}{h};\; -n \le P \le 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

$$(i) \Longrightarrow \qquad y = f(x) = f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots iii)$$

Differentiate with respect to 'x' and using (i) & (ii)

$$y' = \frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx} = \frac{d}{dp}\left[f(x_n) + P\nabla f(x_n) + \frac{P(P+1)}{2!}\nabla^2 f(x_n) + \cdots \ldots\ldots\ldots\right]\frac{d}{dx}\left(\frac{x-x_n}{h}\right)$$

$$y' = \frac{d}{dp}\left[0 + \nabla f(x_n) + \frac{(2P+1)}{2}\nabla^2 f(x_n) + \cdots \ldots\ldots\ldots\right]\left(\frac{1-0}{h}\right)$$

$$y' = \frac{1}{h}\left[\nabla f(x_n) + \frac{(2P+1)}{2}\nabla^2 f(x_n) + \left(\frac{3P^2+6P+2}{6}\right)\nabla^3 f(x_n) + \left(\frac{4P^3+18P^2+22P+6}{24}\right)\nabla^4 f(x_n) \ldots\ldots\ldots\ldots\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

Differentiate y' with respect to 'x'

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dp}.\frac{dp}{dx} = \frac{1}{h}\left[\nabla^2 f(x_n) + (P+1)\nabla^3 f(x_n) + \left(\frac{6P^2+18P+11}{12}\right)\nabla^4 f(x_n) \ldots\ldots\ldots\ldots\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

Equation (iv)& (v) are Newton's backward interpolation formulae which can be used to compute 1st and 2nd derivatives of a tabular function near the end of table similarly

Expression of Newton's forward interpolation formulae can be derived to compute the 1st, 2nd and higher order derivatives near the beginning of table of values.

DIFFERENTIATION USING CENTRAL DIFFERENCE OPERATOR (σ)

Since $\sigma = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Since $hD = \log E \text{ and } E = e$ **therefore** $\sigma = e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}$

Also as $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$ **therefore** $\sigma = 2\sin\left(\frac{hD}{2}\right)$

$$\Rightarrow \frac{\sigma}{2} = \sinh\left(\frac{hD}{2}\right) \quad \Rightarrow \sinh^{-1}\left(\frac{\sigma}{2}\right) = \left(\frac{hD}{2}\right) \quad \Rightarrow D = \frac{2}{h}\ \sinh^{-1}\frac{\sigma}{2}$$

Since by Maclaurin series

$$\sinh^{-1}(x) = x - \frac{1}{2}\left(\frac{x^3}{3}\right) + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots\ldots\ldots\ldots$$

$$\Rightarrow D = \frac{2}{h}\left[\frac{\sigma}{2} - \frac{1}{2}\left(\frac{\left(\frac{\sigma}{2}\right)^3}{3}\right) + \frac{1.3}{2.4}\frac{\left(\frac{\sigma}{2}\right)^5}{5} - \frac{1.3.5}{2.4.6}\frac{\left(\frac{\sigma}{2}\right)^7}{7} + \cdots\right]$$

$$D = \frac{1}{h}\left[\sigma - \frac{\sigma^3}{24} + \frac{3\sigma^5}{640}\ldots\ldots\ldots\ldots\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Similarly, for second derivatives squaring (i) and simplifying

$$D^2 = \frac{1}{h^2}\left[\sigma^2 - \frac{\sigma^4}{12} + \frac{\sigma^6}{90} - \cdots \dots\dots\dots\dots..\right]$$

$$D^2y = y'' = \frac{1}{h^2}\left[\sigma^2 y - \frac{\sigma^4 y}{12} + \frac{\sigma^6 y}{90} - \cdots \dots\dots\dots\dots..\right] \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

For calculating first and second derivative at an inter tabular form (point) we use (i) and (ii) while 1st derivative can be computed by another convergent form for D_i which can derived as follows

Since $D = \frac{1}{h}\left[\sigma - \frac{\sigma^3}{24} + \frac{3\sigma^5}{640} \dots\dots\dots\dots\dots..\right]$

Multiplying R.H.S by $\frac{\mu}{\sqrt{1+\frac{\delta^2}{4}}} = 1$ **which is unity and noting the binomial expansion**

$$\left(1 + \frac{\delta^2}{4}\right)^{-1/2} = 1 - \frac{\sigma^2}{8} + \frac{3\sigma^4}{128} - \frac{15\sigma^6}{48\times 64} \dots\dots\dots\dots\dots\dots\dots.\dots\dots$$

We get

$$D = \frac{\mu}{h}\left[1 - \frac{\sigma^2}{8} + \frac{3\sigma^4}{128} \dots\dots\dots\dots\dots\dots\dots.\dots\dots..\right]\left[\sigma - \frac{\sigma^3}{24} + \frac{3\sigma^5}{640} \dots\dots\dots\dots\dots..\right]$$

$$\Rightarrow D = \frac{\mu}{h}\left[\sigma - \frac{\sigma^3}{68} + \frac{4\sigma^5}{120} \dots\dots\dots\dots\dots..\right]$$

Therefore $\Rightarrow D' = Dy = \frac{\mu}{h}\left[\sigma y - \frac{\sigma^3}{68} y + \frac{4\sigma^5}{120} y \dots\dots\dots\dots\dots..\right] \dots\dots\dots\dots\dots. (iii)$

Equation (ii) and (iii) are called <u>STERLING FORMULAE</u> for computing the derivative of a tabular function. Equation (iii) can also be written as

$$D' = Dy = \frac{\mu}{h}\left[\sigma y - \frac{1^2}{3!}\sigma^3 y + \frac{1^2 2^2}{5!}\sigma^5 y - \frac{1^2 2^2 3^2}{7!}\sigma^7 y + \cdots \dots\dots\dots\dots..\right]$$

STERLING FORMULA

Sterling's formula is

$$y = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}(\Delta^2 y_{-1}) + \frac{p(p^2-1^2)}{3!}\left[\frac{(\Delta^3 y_{-1} - \Delta^3 y_{-2})}{2}\right] + \frac{p^2(p^2-1^2)}{4!}(\Delta^4 y_{-2}) + \cdots \dots....$$

Where $p = \frac{x - x_0}{h}$

TWO AND THREE POINT FORMULAE

Since $$y_i' = \frac{\Delta}{h} y_i = \frac{y_{i+1} - y_i}{h} = \frac{y(x_i + h) - y(x_i)}{h} \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Similarly $$y_i' = \frac{\nabla}{h} y_i = \frac{y_i - y_{i-1}}{h} = \frac{y(x_i) - y(x_i - h)}{h} \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Adding (i) and (ii) we get

$$2y_i' = \frac{y(x_i + h) - y(x_i - h)}{h} \quad \Longrightarrow \quad y_i' = \frac{1}{2h}[y(x_i + h) - y(x_i - h)] \ldots\ldots\ldots\ldots\ldots (iii)$$

Subtracting (i) and (iii) we get two point formulae for the first derivative

Similarly, we know that

$$y_i'' = \frac{\Delta^2}{h^2} y_i = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} = \frac{1}{h^2}[y(x_i + 2h) - 2y(x_i + h) + y(x_i)] \ldots\ldots\ldots\ldots (iv)$$

And $$y''_i = \frac{\nabla^2}{h^2} y_i = \frac{y_i - 2y_{i-1} + y_{-2}}{h^2}$$

$$y''_i = \frac{1}{h}[y(x_i) - 2y(x_i - h) + y(x_i - 2h)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

Similarly

$$y''_i = \frac{\sigma^2}{h^2} y_i = \frac{\sigma y_{i+\frac{1}{2}} - \sigma y_{i-\frac{1}{2}}}{h^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$y''_i = \frac{y(x_i - h) - 2y(x_i) + y(x_i - h)}{h^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (vi)$$

By subtracting (iv) and (vi) we get three point formulae for computing the 2nd derivative.

NUMERICAL INTEGRATION

The process of producing a numerical value for the defining integral $\int_a^b f(x)dx$ is called Numerical Integration. Integration is the process of measuring the Area under a function plotted on a graph. Numerical Integration is the study of how the numerical value of an integral can be found.

Also called Numerical Quadrature if $\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i)$ which refers to finding a square whose area is the same as the area under the curve.

A GENERAL FORMULA FOR SOLVING NUMERICAL INTEGRATION

This formula is also called a general quadrature formula.

Suppose f(x) is given for equidistant value of 'x' say a=x_0, x_0+h,x_0+2h …. x_0+nh = b

Let the range of integration (a,b) is divided into 'n' equal parts each of width 'h' so that "b-a=nh".

By using fundamental theorem of numerical analysis It has been proved the general quadrature formula which is as follows

$$I = h\left[nf(x_0) + \frac{n^2}{2}\Delta f(x_0) + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f(x_0)}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f(x_0)}{3!} + \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11}{3}n^3 - 3n^2\right)\frac{\Delta^4 f(x_0)}{4!} + \cdots \dots\dots\dots\dots\dots.. + up\ to\ (n+1) terms\right]$$

Bu putting n into different values various formulae is used to solve numerical integration.

That are Trapezoidal Rule, Simpson's 1/3, Simpson's 3/8, Boole's, Weddle's etc.

IMPORTANCE: **Numerical integration is useful when**

- **Function cannot be integrated analytically.**
- **Function is defined by a table of values.**
- **Function can be integrated analytically but resulting expression is so complicated.**

COMPOSITE (MODIFIED) NUMERICAL INTEGRATION

Trapezoidal and Simpson's rules are limited to operating on a single interval. Of course, since definite integrals are additive over subinterval, we can evaluate an integral by dividing the interval up into several subintervals, applying the rule separately on each one and then totaling up. This strategy is called Composite Numerical Integration.

TRAPEZOIDAL RULE

Rule is based on approximating $f(x)$ by a piecewise linear polynomial that interpolates $f(x)$ at the nodes "$x_0, x_1, \dots \dots \dots x_n$"

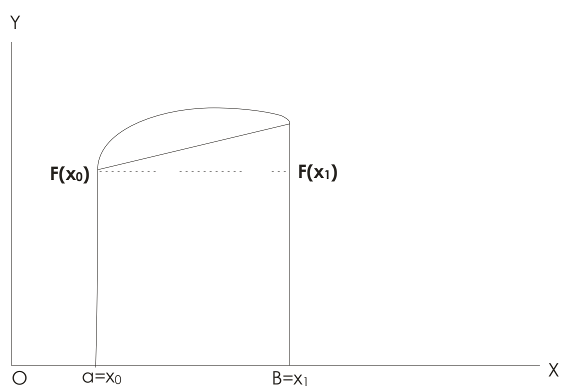
Trapezoidal Rule defined as follows

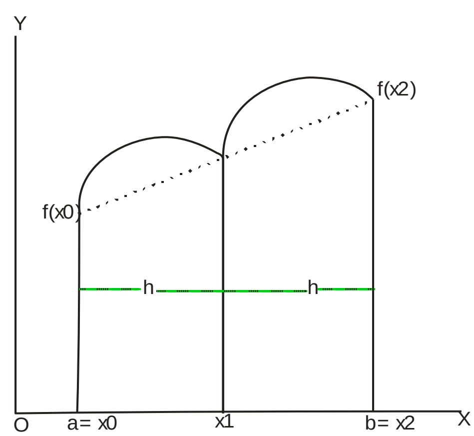
$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{12}y''(a)$ **And this is called <u>Elementary Trapezoidal Rule</u>.**

Composite form of Trapezoidal Rule is $\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \dots . + y_{n-1}) + y_n]$

DARIVATION (1st METHOD)

Consider a curve $y = f(x)$ bounded by $x_0 = a$ and $x_1 = b$ we have to find $\int_a^b f(x)dx$ i.e. Area under the curve $y = f(x)$ then for one Trapezium under the area i.e. n = 1





$$\int_a^b f(x)dx = Area\ of\ Trapezium = \frac{sum\ of\ parallel\ sides}{2} \times perpendicular$$

$$\int_a^b f(x)dx = \frac{f(x_0)+f(x_1)}{2} \times h = \frac{h}{2}[f(x_0) + f(x_1)]$$

For two trapeziums i. e. n = 2

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] = \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$$

For n = 3 $\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \frac{h}{2}[f(x_2) + f(x_3)]$

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2[f(x_1) + f(x_2)] + f(x_3)]$$

In general for n – trapezium the points will be "$x_0, x_1, \dots \dots ... x_n$" and function will be "$y_0, y_1, \dots \dots ... y_n$"

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2[f(x_1) + f(x_2) + \cdots \dots \dots \dots + f(x_{n-1})] + \ f(x_n)]$$

$$\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \dots \dots \dots + y_{n-1}) + y_n]$$

Trapezium rule is valid for n (number of trapezium) is even or odd.

The accuracy will be increase if number of trapezium will be increased OR step size will be decreased mean number of step size will be increased.

DARIVATION (2nd METHOD)

Define y = f(x) in an interval $[a, b] = [x_0, x_n]$ then

$$\int_{x_0}^{x_0} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots \dots \dots \dots \dots .. + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$\int_{x_0}^{x_0} f(x)dx = \left[\frac{h}{2}(y_0 + y_1)\right] + \left[\frac{h}{2}(y_1 + y_2)\right] + \cdots \dots \dots \dots .. + \left[\frac{h}{2}(y_{n-1} + y_n)\right] + \epsilon_n$$

Where $\epsilon_n = -\frac{h^3}{12}[y''(a_1) + y''(a_2) + \cdots \dots \dots \dots \dots + y''(a_n)]$ **is global error.**

$\Rightarrow \epsilon_n = -\frac{h^3}{12}[ny''(a)]$

Therefore $\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots \dots \dots \dots + y_{n-1}) + y_n]$ **Where** $a = x_0 \text{ and } b = x_n$

REMEMBER: The maximum incurred in approximate value obtained by Trapezoidal Rule is nearly equal to $\frac{(b-a)^3 M}{12n^2}$ **where** $M = max|f''(x)| \quad on \ [a, b]$

EXAMPLE: Evaluate $I = \int_0^1 \frac{1}{1+x^2}\text{dx}$ **using Trapezoidal Rule when** $h = ¼$

SOLUTION

X	0	1/4	1/2	3/4	1
F(x)	1	0.9412	0.8000	0.6400	0.5000

Since by Trapezoidal Rule $\int_0^1 \frac{1}{1+x^2}\text{dx} = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = 0.7828$

SIMPSON'S $(\frac{1}{3})$ RULE

Rule is based on approximating f(x) by a Quadratic Polynomial that interpolate f(x) at $x_{i-1}, x_i \text{ and } x_{i+1}$

Simpson's Rule is defined as for simple case $\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] - \frac{h^5}{90}y^{iv}(\S)$

While in composite form it is defined as

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots \dots \dots .. + y_{2N-1}) + 2(y_2 + y_4 + \cdots \dots \dots + y_{2N-2}) + y_N]$$

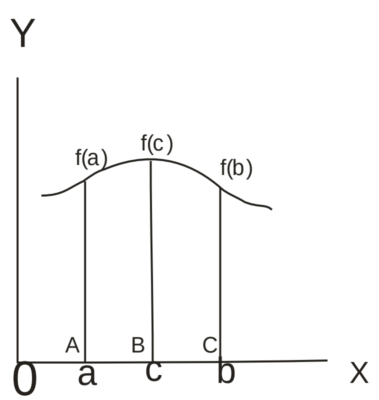
Global error for Simpson's Rule is defined as $\in = -\frac{x_{2N}-x_0}{180}h^4 y^{iv}(\S) = O(h^4)$

REMARK

In Simpson Rule number of trapezium must of Even and number of points must of Odd.

DERIVATION OF SIMPSON'S $(\frac{1}{3})$ RULE (1st method)

Consider a curve bounded by x = a and x = b and let 'c' is the mid-point between a and b such that $a << b$ we have to find $\int_a^b f(x)dx$ i.e. Area under the curve.



Consider $X = C + Y \dots \dots \dots \dots ..(i) \Rightarrow \quad dx = dy$

Now $c = OB = OA + AB \Rightarrow c = a + h \Rightarrow a = c - h$

$b = OC = OB + BC \Rightarrow b = c + h$

$(i) \Rightarrow put\ x = a\ then\ a = c + y \Rightarrow c - h = c + y \Rightarrow -h = y$

$put\ x = b\ then\ c + h = c + y \Rightarrow h = y$

Now $\int_a^b f(x)dx = \int_{-h}^{+h} f(c+y)dy$ **where y is small change**

Using Taylor Series Formula $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots \ldots\ldots\ldots\ldots\ldots$

$$\int_{-h}^{+h} f(c+y)dy = \int_{-h}^{+h}\left[f(c) + yf'(c) + \frac{y^2}{2!}f''(c) + \cdots \ldots\ldots\ldots\ldots..\right]dy$$

Neglecting higher derivatives

$$\int_{-h}^{+h} f(c+y)dy = \int_{-h}^{+h}\left[f(c) + yf'(c) + \frac{y^2}{2!}f''(c)\right]dy$$

$$\int_{-h}^{+h} f(c+y)dy = \left|yf(c) + \frac{y^2}{2}f'(c) + \frac{y^3}{2.3}f''(c)\right|_{-h}^{h} = 2h\left[f(c) + \frac{h^2}{6}f''(c)\right] \ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

$$f(a) = f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!}f''(c) + neglected$$

$$f(b) = f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + neglected$$

$$f(c-h) - f(c+h) = 2f(c) + 2\frac{h^2}{2!}f''(c)$$

$f(c-h) - f(c+h) - 2f(c) = h^2 f''(c)$ **Put this value in (i)**

$$\int_a^b f(x)dx = 2h[f(c) + \frac{1}{6}\{f(c-h) + f(c+h) - 2f(c)\}]$$

$$\int_a^b f(x)dx = \frac{2h}{6}[6f(c) + f(c-h) + f(c+h) - 2f(c)]$$

$$\int_a^b f(x)dx = \frac{h}{3}[4f(c) + f(c-h) + f(c+h)] = \frac{h}{3}[4f(c) + f(a) + f(b)]$$

$$\int_a^b f(x)dx = \frac{h}{3}[4f(x_1) + f(x_0) + f(x_2)] = \frac{h}{3}[4y_1 + y_0 + y_2]$$

For n = 4

$$\int_{x_0}^{x_4} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\int_{x_0}^{x_4} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3) + 2y_2 + y_4]$$

In General

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 \ldots\ldots..y_{2N-1}) + 2(y_2 + y_4 \ldots\ldots + y_{2N-2}) + y_{2N}]$$

DERIVATION OF SIMPSON'S $(\frac{1}{3})$ RULE (2nd method)

$$\int_{x_0}^{x_{2N}} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots \ldots \ldots \ldots \ldots \ldots . \ldots \ldots + \int_{x_{2N-2}}^{x_{2N}} f(x)dx$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4] + \cdots \ldots \ldots \ldots \ldots \ldots .. + \frac{h}{3}[y_{2N-2} + 4y_{2N-1} + y_{2N}]$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 \ldots \ldots y_{2N-1}) + 2(y_2 + y_4 \ldots + y_{2N-2}) + y_{2N}]$$

This is required formula for Simpson's (1/3) Rule

EXAMPLE

Compute $I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-\frac{x^2}{2}} \mathrm{dx}$ **using Simpson's (1/3) Rule when** $h = 0.125$

SOLUTION

X	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1
F(x)	0.798	0.792	0.773	0.744	0.704	0.656	0.602	0.544	0.484

Since by Simpson's Rule

$$\sqrt{\frac{2}{\pi}} \int_0^1 e^{-\frac{x^2}{2}} \mathrm{dx} = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$\sqrt{\frac{2}{\pi}} \int_0^1 e^{-\frac{x^2}{2}} \mathrm{dx} = 0.6827$ **After putting the values.**

SIMPSON'S $(\frac{3}{8})$ RULE

Rule is based on fitting four points by a cubic.

Simpson's Rule is defined as for simple case

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] - \frac{3h^5}{80} y^{iv}(\S)$$

While in composite form ("n" must be divisible by 3) it is defined as

$$\int_{x_0}^{x_N} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + \cdots \dots \dots .. + y_{N-1}) + 2(y_3 + y_6 + \cdots \dots \dots .. + y_{N-3}) + y_N]$$

DERIVATION

$$\int_{x_0}^{x_N} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots \dots \dots \dots \dots \dots . \dots \dots + \int_{x_{N-3}}^{x_N} f(x)dx$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] + \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

$$+ \cdots \dots \dots \dots \dots \dots .. + \frac{3h}{8}[y_{N-3} + 3y_{N-2} + 3y_{N-1} + y_N]$$

$$\int_{x_0}^{x_N} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + \cdots \dots \dots .. + y_{N-1}) + 2(y_3 + y_6 + \cdots \dots \dots .. + y_{N-3}) + y_N]$$

This is required formula for Simpson's (3/8) Rule.

REMARK: Global error in Simpson's (1/3) and (3/8) rule are of the same order but if we consider the magnitude of error then Simpson (1/3) rule is superior to Simpson's (3/8) rule.

TRAPEZOIDAL AND SIMPSON'S RULE ARE CONVERGENT

If we assume Truncation error, then in the case of Trapezoidal Rule

$I - A = -\frac{(b-a)h^2}{12} y^2(\S)$ **Where "I" is the exact integral and "A" the approximation. If "$lmt\ h = 0$" then assuming "y^2" bounded**

"$lmt\ (I - A) = 0$" **(This the definition of convergence of Trapezoidal Rule)**

For Simpson's Rule we have the similar result

$I - A = -\frac{(b-a)h^4}{180} y^4(\S)$

If "$lmt\ h = 0$" then assuming "y^4" bounded

"$lmt\ (I - A) = 0$" **(This the definition of convergence of Simpson's Rule)**

ERROR TERMS

Rectangular Rule	$\frac{h^2}{2!} y'(\S)$	$x_0 < \S < x_1$
Trapezoidal Rule	$-\frac{h^3}{12} y''(\S)$	$x_0 < \S < x_1$
Simpson's (1/3) Rule	$-\frac{h^5}{90} y^{iv}(\S)$	$x_0 < \S < x_1$
Simpson's (3/8) Rule	$-\frac{3h^5}{80} y^{iv}(\S)$	$x_0 < \S < x_1$

WEDDLE'S

In this method "n" should be the multiple of 6. Rather function will not applicable. This method also called sixth order closed Newton's cotes (or) the first step of Romberg integration.

First and last terms have no coefficients and other move with 5, then 1, then 6.

Weddle's Rule is given by formula

$$\int_a^b f(x)dx = \frac{3h}{10}\begin{bmatrix} f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + \cdots \ldots \ldots \ldots + \\ + \cdots \ldots . . +5f(x_{n-3}) + f(x_{n-2}) + 6f(x_{n-1}) + f(x_n) \end{bmatrix}$$

EXAMPLE: for $\int_{0.25}^{1.75} \frac{1}{1+x^2} \mathrm{dx}$ at n = 6

X	0.25	0.5	0.75	1	1.25	1.5	1.75
F(x)	0.9411	0.8	0.64	0.5	0.4	0.3	0.2

Now using formula $\int_{0.25}^{1.75} \frac{1}{1+x^2} \mathrm{dx} = \frac{3(0.25)}{10}[y_0 + 5y_1 + y_2 + 6y_3 + 5y_4 + y_5 + y_6] = 0.8310$

BOOLE'S RULE

The method approximate $\int_{x_0}^{x_4} f(x)\mathrm{dx}$ for '5' equally spaced values. Rule is given by George Bool. Rule is given by following formula

$$\int_a^b f(x)\mathrm{dx} = \frac{2\mathrm{h}}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

EXAMPLE: Evaluate $\int_{0.2}^{0.6} \frac{1}{1+x^2} \mathrm{dx}$ at n = 4 and h = 0.1

SOLUTION

X	0.2	0.3	0.4	0.5	0.6
F(x)	0.96	0.92	0.86	0.80	0.74

Now using formula $\int_{0.2}^{0.6} \frac{1}{1+x^2} \mathrm{dx} = \frac{2(0.1)}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$

$\int_{0.2}^{0.6} \frac{1}{1+x^2} \mathrm{dx} = 0.3399$ After putting the values.

RECTANGULAR RULE

Rule is also known as Mid-Point Rule. And is defined as follows for ‘n + 1’ points.

$$\int_a^b f(x)\mathrm{dx} = \mathrm{h}[f(x_0) + f(x_1) + \cdots \ldots \ldots \ldots \ldots . + f(x_n)]$$

In general $\int_a^b f(x)\mathrm{dx} = \mathrm{h}\sum_{i=0}^{n} f(x_i)$

REMEMBER

- **As we increased ‘n’ or decreased ‘h’ the accuracy improved and the approximate solution becomes closer and closer to the exact value.**
- **If ‘n’ is given, then use it. If ‘h’ is given, then we can easily get ‘n’.**
- **If ‘n’ is not given and only ‘points’ are discussed, then ‘1’ less that points will be ‘n’. For example, if ‘3’ points are given then ‘n’ will be ‘2’.**
- **If only table is given, then by counting the points we can tell about ‘n’.one point will be greater than ‘n’ in table.**

EXAMPLE

Evaluate $\int_1^3 \frac{1}{x^2}\mathrm{dx}$ **for n = 4 using Rectangular Rule.**

SOLUTION

Here a = 1, b = 3 then $h = \frac{b-a}{n} = 0.5$

X	1	3/2	2	5/2	3
F(x)	1	4/9	1/4	4/25	1/5

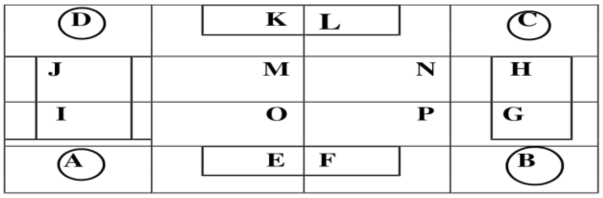
$\mathit{Now\ using\ formula}$ $\int_1^3 \frac{1}{x^2}\mathrm{dx} = \mathrm{h}[f(x_0) + f(x_1) + f(x_3)] = 0.925$

DOUBLE INTEGRATION

Double Integral

Trapezoidal Rule

Evaluate $\int_c^d \int_a^b f(x, y)\,dx\,dy$ where a, b, c, d are constants.



I= $\frac{hk}{4}$ **{[sum of values in]+2(sum of values in ▭ +4[sum of remaining values]}**

Simpson's Rule

$$I = \frac{hk}{9}\left\{\begin{array}{l} \text{(sum of the values of f at four corners)} \\ +2\text{(sum of the values of f at the odd positions on the} \\ \text{boundary except the corners)} \\ +4\text{(sum of the values of f at the even positions on the boundary)} \\ +\{4\text{(sum of the values of f at the odd positions)}+ \\ 8\text{(sum of the values of f at the even positions)} \\ \text{on the odd row f of the matrix except boundary rows}\}+ \\ \{8\text{(sum of the values of f at the odd positions)}+ \\ 16\text{(sum of the values of f at the even positions)} \\ \text{on the even row f of the matrix}\} \end{array}\right\}$$

Problems based on Double integrals

1. Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy$ using Trapezoidal and Simpson's rule. Verify your result by actual integration.

Solution:

Divide the range of x and y into 4 equal parts

$$h = \frac{2.4-2}{4} = 0.1$$

$$k = \frac{1.4-1}{4} = 0.1$$

Get the values of f(x, y) = $\frac{1}{xy}$ at nodal points

Y/X	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

Now using previous formulae we get the required results

***FOR TRAPEZOIDAL RULE*:** $I = 0.0614$ ***FOR SIMPSON'S RULE*:** $I = 0.0613$

Verify actual integration by yourself.

QUESTION: Evaluate $\int_1^2 \int_1^2 \frac{dxdy}{x+y}$ by Trapezoidal rule for h = 0.25 = k

SOLUTION: $1 \le x \le 2 \implies x_0 = 1, x_1 = x_0 + h = 1.25, x_2 = 1.50, x_3 = 1.75, x_4 = 2$

And $1 \le y \le 2 \implies y_0 = 1, y_1 = y_0 + k = 1.25, y_2 = 1.50, y_3 = 1.75, y_4 = 2$

STEP – I: $f(x, y) = \frac{1}{x+y}$

Y/X	1	1.25	1.50	1.75	2
1	$\frac{1}{1+1}$ =0.5	0.4444	0.4	0.3636	0.3333
1.25	0.4444	0.4	0.3636	0.3333	0.3077
1.50	0.4	0.3636	0.3333	0.3077	0.2857
1.75	0.3636	0.3333	0.3077	0.2857	0.2667
2	0.3333	0.3077	0.2857	0.2667	0.25

STEP –I I:

$$I_1 = \int_1^2 f(1, y)dy = \frac{k}{2}[f(1, y_0) + f(1, y_4) + 2[f(1, y_1) + f(1, y_2) + f(1, y_3)] = 0.4062$$

$$I_2 = \int_1^2 f(1.25, y)dy = \frac{k}{2}[f(1.25, y_0) + f(1.25, y_4) + 2[f(1.25, y_1) + f(1.25, y_2) + f(1.25, y_3)] = 0.3682$$

$$I_3 = \int_1^2 f(1.5, y)dy = \frac{k}{2}[f(1.5, y_0) + f(1.5, y_4) + 2[f(1.5, y_1) + f(1.5, y_2) + f(1.5, y_3)] = 0.3369$$

$$I_4 = \int_1^2 f(1.75, y)dy = \frac{k}{2}[f(1.75, y_0) + f(1.75, y_4) + 2[f(1.75, y_1) + f(1.75, y_2) + f(1.75, y_3)] = 0.3105$$

$$I_5 = \int_1^2 f(2, y)dy = \frac{k}{2}[f(2, y_0) + f(2, y_4) + 2[f(2, y_1) + f(2, y_2) + f(2, y_3)] = 0.2879$$

STEP –III:

$$I = \int_1^2 \int_1^2 \frac{dxdy}{x+y} = \frac{h}{2}[I_1 + I_5 + 2(I_2 + I_3 + I_4)] = 0..3407$$

QUESTION: Evaluate $\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{sin(x+y)}\,dxdy$

SOLUTION: Take n = 4 (by own choice) then $h=\frac{b-a}{n}=\frac{\frac{\pi}{2}-0}{4}=\frac{\pi}{8}=k\ (also)$

$0\le x\le\frac{\pi}{2}\Rightarrow x_0=0, x_1=x_0+h=\frac{\pi}{8}, x_2=\frac{\pi}{4}, x_3=\frac{3\pi}{8}, x_4=\frac{\pi}{2}$

And $1\le y\le\frac{\pi}{2}\Rightarrow y_0=0, y_1=y_0+k=\frac{\pi}{8}, y_2=\frac{\pi}{4}, y_3=\frac{3\pi}{8}, y_4=\frac{\pi}{2}$

STEP – I: $f(x,y)=\sqrt{sin(x+y)}$

Y/X	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
0	0	0.6186	0.8409	0.9612	1
$\frac{\pi}{8}$	0.6186	0.8409	0.9612	1	0.9612
$\frac{\pi}{4}$	0.8409	0.9612	1	0.9612	0.8409
$\frac{3\pi}{8}$	0.9612	1	0.9612	0.8409	0.6186
$\frac{\pi}{2}$	1	0.9612	0.8409	0.6186	0

STEP –I I:

$$I_1=\int_0^{\frac{\pi}{2}} f(0,y)dy=\frac{k}{2}[f(0,y_0)+f(0,y_4)+2[f(0,y_1)+f(0,y_2)+f(0,y_3)]=1.1469$$

$$I_2=\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{8},y\right)dy=\frac{k}{2}[f\left(\frac{\pi}{8},y_0\right)+f\left(\frac{\pi}{8},y_4\right)+2\left[f\left(\frac{\pi}{8},y_1\right)+f\left(\frac{\pi}{8},y_2\right)+f\left(\frac{\pi}{8},y_3\right)\right]=1.4106$$

$$I_3=\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{4},y\right)dy=\frac{k}{2}[f\left(\frac{\pi}{4},y_0\right)+f\left(\frac{\pi}{4},y_4\right)+2\left[f\left(\frac{\pi}{4},y_1\right)+f\left(\frac{\pi}{4},y_2\right)+f\left(\frac{\pi}{4},y_3\right)\right]=1.4778$$

$$I_4=\int_0^{\frac{\pi}{2}} f\left(\frac{3\pi}{8},y\right)dy=\frac{k}{2}[f\left(\frac{3\pi}{8},y_0\right)+f\left(\frac{3\pi}{8},y_4\right)+2\left[f\left(\frac{3\pi}{8},y_1\right)+f\left(\frac{3\pi}{8},y_2\right)+f\left(\frac{3\pi}{8},y_3\right)\right]=1.4106$$

$$I_5=\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2},y\right)dy=\frac{k}{2}[f\left(\frac{\pi}{2},y_0\right)+f\left(\frac{\pi}{2},y_4\right)+2\left[f\left(\frac{\pi}{2},y_1\right)+f\left(\frac{\pi}{2},y_2\right)+f\left(\frac{\pi}{2},y_3\right)\right]=1.1469$$

STEP –III: $I=\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{sin(x+y)}\,dxdy=\frac{h}{2}[I_1+I_5+2(I_2+I_3+I_4)]=2.1386$

QUESTION: **Evaluate** $\iint_D \frac{dxdy}{x^2+y^2}$ **where D is the square with comes at (1,1) , (2,1) ,(2,2) , (1,2)**

SOLUTION: **Take n = 4 (by own choice) then** $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8} = k\ (also)$

$1 \le x \le 2$ **Also** $1 \le y \le 2$

STEP – I: $f(x,y) = \frac{1}{x^2+y^2}$

Y/X	1	1.25	1.50	1.75	2
1	0.5	0.3902	0.3077	0.2462	0.2
1.25	0.3902	0.3200	0.2623	0.2162	0.1798
1.50	0.3077	0.2623	0.2222	0.1882	0.1600
1.75	0.2462	0.2162	0.1882	0.1633	0.1416
2	0.2	0.1798	0.1600	0.1416	0.1250

STEP –I I:

$$I_1 = \int_1^2 f(1,y)dy = \frac{k}{2}[f(1,y_0) + f(1,y_4) + 2[f(1,y_1) + f(1,y_2) + f(1,y_3)] = 0.3235$$

$$I_2 = \int_1^2 f(1.25,y)dy = \frac{k}{2}[f(1.25,y_0) + f(1.25,y_4) + 2[f(1.25,y_1) + f(1.25,y_2) + f(1.25,y_3)] = 0.2709$$

$$I_3 = \int_1^2 f(1.5,y)dy = \frac{k}{2}[f(1.5,y_0) + f(1.5,y_4) + 2[f(1.5,y_1) + f(1.5,y_2) + f(1.5,y_3)] = 0.2266$$

$$I_4 = \int_1^2 f(1.75,y)dy = \frac{k}{2}[f(1.75,y_0) + f(1.75,y_4) + 2[f(1.75,y_1) + f(1.75,y_2) + f(1.75,y_3)] = 0.1904$$

$$I_5 = \int_1^2 f(2,y)dy = \frac{k}{2}[f(2,y_0) + f(2,y_4) + 2[f(2,y_1) + f(2,y_2) + f(2,y_3)] = 0.1610$$

STEP –III:

$$I = \int_1^2 \int_1^2 \frac{dxdy}{x^2+y^2} = \frac{h}{2}[I_1 + I_5 + 2(I_2 + I_3 + I_4)] = 0.2325$$

QUESTION: Evaluate $\int_0^1 \int_1^2 (x^2 + y^2)dxdy$ **by using Simpson (1/3) rule**

SOLUTION: Take n = 4 (by own choice) then $k = \frac{b-a}{n} = \frac{1-0}{4} = 0.25 = h\ (also)$

$1 \le x \le 2$ **Also** $0 \le y \le 1$

STEP – I: $f(x, y) = x^2 + y^2$

Y/X	0	0.25	0.50	0.75	1
1	1	1.6250	1.25	1.5625	2
1.25	1.5625	1.6250	1.8125	2.1250	2.5625
1.50	2.25	2.3125	2.5	2.8125	3.25
1.75	3.0625	3.1250	3.3125	3.6250	4.0625
2	4	4.0625	4.2500	4.5625	5

STEP –I I:

$$I_1 = \int_0^1 f(1, y)dy = \frac{k}{3}[f(1, y_0) + f(1, y_4) + 2f(1, y_2) + 4[f(1, y_1) + f(1, y_3)] = 1.3333$$

$$I_2 = \int_0^1 f(1.25, y)dy = \frac{k}{3}[f(1.25, y_0) + f(1.25, y_4) + 2f(1.25, y_2) + 4[f(1.25, y_1) + f(1.25, y_3)] = 1.8958$$

$$I_3 = \int_0^1 f(1.50, y)dy = \frac{k}{3}[f(1.50, y_0) + f(1.50, y_4) + 2f(1.50, y_2) + 4[f(1.50, y_1) + f(1.50, y_3)] = 2.5832$$

$$I_4 = \int_0^1 f(1.75, y)dy = \frac{k}{3}[f(1.75, y_0) + f(1.75, y_4) + 2f(1.75, y_2) + 4[f(1.75, y_1) + f(1.75, y_3)] = 3.3958$$

$$I_5 = \int_0^1 f(2, y)dy = \frac{k}{3}[f(2, y_0) + f(2, y_4) + 2f(2, y_2) + 4[f(2, y_1) + f(2, y_3)] = 4.3316$$

STEP –III:

$$I = \int_0^1 \int_1^2 (x^2 + y^2)dxdy = \frac{h}{2}[I_1 + I_5 + 2I_3 + 4(I_2 + I_4)] = 2.6654$$

GUASSIAN QUADRATURE FORMULAE

DERIVATION OF TWO-POINT GAUSS QUADRATURE RULE

Method 1:

The two-point Gauss quadrature rule is an extension of the trapezoidal rule approximation where the arguments of the function are not predetermined as *a* and *b,* but as unknowns x_1 and x_2. So in the two-point Gauss quadrature rule, the integral is approximated as

$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$

There are four unknowns x_1, x_2, c_1 and c_2. These are found by assuming that the formula gives exact results for integrating a general third order polynomial,

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$

Hence $\int_a^b f(x)dx = \int_a^b (a_0 + a_1x + a_2x^2 + a_3x^3)dx$

$\int_a^b f(x)dx = \left|(a_0\mathrm{x} + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4})\right|_a^b$

$\int_a^b f(x)dx = \left[a_0(\mathrm{b}-\mathrm{a}) + a_1(\frac{b^2-a^2}{2}) + a_2(\frac{b^3-a^3}{3}) + a_3(\frac{b^4-a^4}{4})\right] \ldots\ldots\ldots\ldots..(i)$

The formula would then give

$\int_a^b f(x)dx \approx c_1f(x_1) + c_2f(x_2) \approx c_1 f(x_1) + c_2 f(x_2)$

$= c_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3) + c_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3)$

$\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

Equating Equations (i) and (ii) gives

$$\begin{bmatrix} a_0(\mathrm{b}-\mathrm{a}) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right) \\ = c_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3) + c_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3) \end{bmatrix}$$

This will give us

$\int_a^b f(x)dx = a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2(c_1x_1^2 + c_2x_2^2) + a_3(c_1x_1^3 + c_2x_2^3)$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(iii)$

Since in Equation (iii), the constants a_0, a_1, a_2, and a_3 are arbitrary, the coefficients of a_0, a_1, a_2, and a_3 are equal. This gives us four equations as follows

$$(iv)\ldots\ldots\ldots\ldots\ldots\ldots. \begin{cases} (\mathrm{b}-\mathrm{a}) = (c_1 + c_2) \\ \left(\frac{b^2-a^2}{2}\right) = (c_1x_1 + c_2x_2) \\ \left(\frac{b^3-a^3}{3}\right) = (c_1x_1^2 + c_2x_2^2) \\ \left(\frac{b^4-a^4}{4}\right) = (c_1x_1^3 + c_2x_2^3) \end{cases}$$

we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$c_1 = \frac{b-a}{2},\ c_2 = \frac{b-a}{2},\ x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2},\ x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

Hence

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) = \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right] + \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right]$$

Method 2

We can derive the same formula by assuming that the expression gives exact values for the individual integrals of $\int_a^b 1dx$, $\int_a^b xdx$, $\int_a^b x^2dx$, and $\int_a^b x^3dx$. The reason the formula can also be derived using this method is that the linear combination of the above integrands is a general third order polynomial given by $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

These will give four equations as follows

$$\begin{cases} \int_a^b 1dx = (\mathrm{b} - \mathrm{a}) = (c_1 + c_2) \\ \int_a^b xdx = \left(\frac{b^2-a^2}{2}\right) = (c_1x_1 + c_2x_2) \\ \int_a^b x^2dx = \left(\frac{b^3-a^3}{3}\right) = (c_1x_1^2 + c_2x_2^2) \\ \int_a^b x^3dx\,.= \left(\frac{b^4-a^4}{4}\right) = (c_1x_1^3 + c_2x_2^3) \end{cases}$$

These four simultaneous nonlinear equations can be solved to give a single acceptable solution

$$c_1 = \frac{b-a}{2},\ c_2 = \frac{b-a}{2},\ x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2},\ x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

Hence $\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) = \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right] + \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right]$

Since two points are chosen, it is called the two-point Gauss quadrature rule. Higher point versions can also be developed.

Higher point Gauss quadrature formulas

For example

$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$ **is called the three-point Gauss quadrature rule. The coefficients c_1, c_2 and c_3, and the function arguments x_1, x_2 and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial**

$\int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5)dx$

General *n* -point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots + c_n f(x_n)$$

A number of particular types of Gaussian formulae are given as follows.

GUASSIAN LEGENDER FORMULA

This formula takes the form $\int_a^b f(x)\mathrm{dx} = \sum_1^n A_i f(x_i)$

And Truncation error for formula is $E = \frac{1}{2n+1}[f(1) + f(-1) + I - \sum_1^n A_i x_i f'(x_i)$

Where "I" is the approximate integral obtained by n – point formula.

GUASS – LAGURRE FORMULA

This formula takes the form $\int_0^\infty e^{-x} f(x)\mathrm{dx} = \sum_1^n A_i f(x_i)$

GUASS – HERMITE FORMULA

This formula takes the form $\int_{-\infty}^\infty e^{-x^2} f(x)\mathrm{dx} = \sum_1^n A_i f(x_i)$

GUASS – CHEBYSHEV FORMULA

This formula takes the form $\int_{-1}^1 \frac{f(x)}{\sqrt{1-X^2}}\mathrm{dx} = \frac{\pi}{n}\sum_1^n f(x_i)$

Where "x_i" is zero n – Chebysheves polynomial

NEWTON'S COTES FORMULA

A quadrature formula of the form $\int_a^b f(x)\mathrm{dx} \approx \sum_0^n C_i f(x_i)$ is called a Newton's Cotes Formula if the nodes "$x_0, x_1, \dots \dots \dots, x_n$" are equally spaced. Where

$$C_i = \int_a^b L_i(x)\mathrm{dx} = \int_a^b \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_i}{x_i-x_j}\mathrm{dx}$$

General Newton's Cotes Formula has the form

$$\int_a^b f(x)\mathrm{dx} = h\sum_0^n f(x_i)\int_0^n \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t-j}{i-j}\mathrm{dt} + \frac{1}{(n+1)!}\int_a^b f^{n+1}(\S_x)\prod_{i=0}^{n}(x-x_i)\,\mathrm{dx}$$

REMARK: Trapezoidal and Simpson's Rule Are Close Newton Cotes formulae while Rectangular Rule is Open Newton Cotes formula.

LIMITATION OF NEWTON'S COTES

Newton's Cotes formulae (Simpson's, Rectangular Rule, and Trapezoidal Rule) are not suitable for Numerical integration over large intervals. Also Newton's Cotes formulas which are based on polynomial interpolation would be inaccurate over a large interval because of oscillatory nature of high degree polynomials. To solve this problem, we use composite Numerical integration.

FORMULA DARIVATION

We shall approximate the given tabulated function by a polynomial "$P_n(x)$" and then integrate this polynomial.

Suppose we are given the data (x_i, y_i) ; $i = 0, 1, 2, \dots \dots . n$ at equispaced points with spacing $h = x_{i+1} - x_i$ we can represent the polynomial by any standard interpolation polynomial.

Now by using Lagrange's formula $f(x) = \sum_0^n l_k(x)y_k$ $\dots \dots \dots \dots \dots \dots \dots ..(i)$

With associated error term $E(x) = \frac{\Pi(x)}{(n+1)!}y^{n+1}(\S)$ $\dots \dots \dots \dots \dots \dots \dots .(ii)$

And $l_k(x) = \frac{\Pi(x)}{(x-x_k)\Pi'(x)}$ $\dots \dots \dots \dots \dots \dots \dots \dots \dots (iii)$

Where $\Pi(x) = (x-x_0)(x-x_1)\dots \dots \dots \dots \dots \dots (x-x_n)$ $\dots \dots \dots \dots \dots \dots \dots (iv)$

Integrating (i) from $a = x_0\ to\ b = x_n$ w. r. to 'x'

$$\int_a^b f(x)\,dx = \int_a^b \sum_0^n l_k(x) y_k\,dx = \int_a^b [l_0(x)y_0 + l_1(x)y_1 + \cdots \dots \dots + l_k(x)y_k]dx$$

$$\int_a^b f(x)\,dx = \sum_0^n \int_a^b l_k(x) y_k\,dx = \sum_0^n \left(\int_a^b l_k(x)\right) y_k dx = \sum_0^n C_k y_k \ \ where\ C_k = \int_a^b l_k(x)\,dx$$

And "C_k" are called Newton's Cotes $\dots \dots \dots \dots \dots \dots \dots \dots \dots .(v)$

HOW TO FIND NEWTON'S COTES?

Let equispaced nodes are defined as $a = x_0\ \ to\ \ b = x_n\ \ and\ \ h = \frac{b-a}{n}$ and $x_k = x_0 + kh$ change the variable $x = x_0 + ph$

Since $a = x_0 = x_0 + 0h, x_1 = x_0 + 1h,$ …………….. $b = x_n = x_0 + nh$ And $x = x_0 + ph$

Using above values in (IV) we get

$$\Pi(x) = (x_0 + ph - x_0)(x_0 + ph - x_1) \dots \dots \dots \dots \dots \dots (x_0 + ph - x_n)$$

$$\Pi(x) = ph[x_0 + ph - (x_0 + h)][x_0 + ph - (x_0 + 2h)] \dots \dots \dots \dots \dots [x_0 + ph - (x_0 + nh)]$$

$$\Pi(x) = ph(ph - h)(ph - 2h) \dots \dots \dots \dots \dots \dots \dots \dots (ph - nh)$$

$$\Pi(x) = h^{n+1}.p(p-1)(p-2) \dots \dots \dots \dots \dots \dots \dots \dots (p-n) \qquad \dots \dots \dots \dots \dots ..(vi)$$

So $l_k(x) = \frac{(x-x_0)(x-x_1)\dots\dots(x-x_{k-1})(x-x_{k+1})\dots\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots\dots(x_k-x_n)}$

Now $x_k = x_0 + kh$ and $x_p = x_0 + ph \quad \Longrightarrow x_k - x_p = (k-p)h$

When

$$p = 0 \Longrightarrow x_k - x_0 = (k-0)h = kh$$

$$p = 1 \Longrightarrow x_k - x_1 = (k-1)h$$

$$\vdots \quad \vdots = \vdots$$

$$\vdots \quad \vdots = \vdots$$

$$p = k-1 \Longrightarrow x_k - x_{k-1} = h$$

$$p = k+1 \Longrightarrow x_k - x_{k+1} = -h$$

$$\vdots \quad \vdots = \vdots$$

$$p = n \Longrightarrow x_k - x_n = (k-n)h = -(n-k)h$$

If x = b, x_0 = a
$\frac{x - x_0}{h} = p$
$\frac{x - x_0}{\frac{b-a}{n}} = p$
n = p

Now putting in "$l_k(x)$" we get

$$l_k(x) = \frac{(x_0+ph-x_0)(x_0+ph-x_0-h)\ldots\ldots\ldots\ldots(x_0+ph-x_0-nh)}{(kh)(k-1)h(k-2)h\ldots\ldots\ldots\ldots\ldots h.(-h)(-2h)[-(n-k)h]}$$

$$l_k(x) = \frac{hp.h(p-1)h(p-2)\ldots\ldots.h[p-(k-1)]h[p-(k+1)]\ldots\ldots.h(p-n)}{(hk)h(k-1)h(k-2)\ldots..h[k-(k-1)]h[k-(k+1)]\ldots..h(k-n)}$$

$$l_k(x) = \frac{h^n.p(p-1)(p-2)\ldots\ldots\ldots.[p-k+1][p-k-1]\ldots\ldots\ldots.(p-n)}{h^n.[k(k-1)(k-2)\ldots\ldots\ldots\ldots.2.1].(-1)^{n-k}[1.2\ldots\ldots\ldots.(n-k)]}$$

$$l_k(x) = \frac{p(p-1)(p-2)\ldots\ldots\ldots.[p-k+1][p-k-1]\ldots\ldots\ldots.(p-n)}{k!(-1)^{n-k}(n-k)!}$$

$$l_k(x) = \frac{p(p-1)(p-2)\ldots\ldots\ldots.[p-k+1][p-k-1]\ldots\ldots\ldots.(p-n)}{k!(-1)^{n-k}(n-k)!} \times \frac{(-1)^{n-k}}{(-1)^{n-k}}$$

$$l_k(x) = \frac{(-1)^{n-k}.p(p-1)(p-2)\ldots\ldots\ldots.[p-k+1][p-k-1]\ldots\ldots\ldots.(p-n)}{k!(-1)^{2(n-k)}(n-k)!}$$

$$l_k(x) = \frac{(-1)^{n-k}.p(p-1)(p-2)\ldots\ldots\ldots.[p-k+1][p-k-1]\ldots\ldots\ldots.(p-n)}{k!(n-k)!} \qquad \ldots\ldots\ldots\ldots\ldots.(vii)$$

Since $C_k = \int_a^b l_k(x)\, dx$ therefore after putting "$l_k(x)$" and "dx"

As "$x = x_0 + ph$" then dx = hdp and if $x \to a$ then $p \to 0$ also $x \to b$ then $p \to n$

$$C_k = \frac{(-1)^{n-k}}{k!(n-k)!}\int_0^n p(p-1)(p-2)\ldots\ldots(p-k+1)(p-k-1)\ldots\ldots(p-n).hdp$$

$$C_k = \frac{(-1)^{n-k}.h}{k!(n-k)!}\int_0^n p(p-1)(p-2)\ldots\ldots(p-k+1)(p-k-1)\ldots\ldots(p-n)\, dp$$

This is required formula for Newton Cotes.

ERROR TERM **let** $\in(x) = \frac{\Pi(x)}{(n+1)!} y^{n+1}(\S) \ldots\ldots\ldots\ldots.\ldots\ldots..(A)$

$$\Pi(x) = h^{n+1}.p(p-1)(p-2)\ldots..(p-n) \qquad \ldots.\ldots\ldots.. \quad (B)$$

Using (B) in (A) we get $\in(x) = \frac{h^{n+1}.p(p-1)\ldots\ldots\ldots\ldots.(p-n).y^{n+1}(\S)}{(n+1)!}$

Integrating both sides $\int_a^b \in(x)dx = \int_o^n \frac{h^{n+1}.p(n+1).p(p-1)\ldots\ldots\ldots\ldots\ldots..(p-n).y^{n+1}(\S)}{(n+1)!}\ hdp$

$$E(x) = \frac{h^{n+2}y^{n+1}(\S)}{(n+1)!}\int_0^n p(p-1)(p-2)\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots(p-n)dp$$

E(x) is called <u>integral error</u>.

ALTERNATIVE METHOD FOR DARIVATION OF TRAPEZOIDAL RULE AND ITS ERROR TERM

$$f(x) = \sum_{k=0}^{n} l_k(x) y_k + \frac{\Pi(x)}{(n+1)!} y^{n+1}(\S)$$

For trapezoidal rule put $n = 1$ $\quad f(x) = \sum_{k=0}^{1} l_k(x) y_k + \frac{\Pi(x)}{2!}\; y''(\S)$

$$f(x) = l_o y_o + l_1 y_1 + \frac{(x-x_0)(x-x_1)}{2}\; y''(\S)$$

Integrating both sides

$$\int_{a=x_0}^{b=x_n=x_1} f(x)dx = y_o \int_{x_0}^{x_1} l_o(x)dx + y_1 \int_{x_0}^{x_1} l_1(x)dx + \frac{y''(\S)}{2} \int_{x_0}^{x_1} (x-x_0)(x-x_1)dx$$

$$\int_{x_0}^{x_1} f(x)dx = y_0 \int_{x_0}^{x_1} \frac{(x-x_1)}{(x_0-x_1)} dx + y_1 \int_{x_0}^{x_1} \frac{(x-x_0)}{(x_1-x_0)} dx + \frac{y''(\S)}{2} \int_{x_0}^{x_1} (x-x_0)(x-x_1)dx$$

Now by changing variables

$$x = x_0 + ph \quad then\; x \to x_0 \Rightarrow p \to 0 \;\; and \;\; x_1 = x_0 + 1h\; then\; x \to x_1 \Rightarrow p \to 1$$

$$\int_{x_0}^{x_1} f(x)dx = y_0 \int_0^1 \frac{(x_0+ph)-(x_0+1h).hdp}{x_0-(x_0+1h)} + y_1 \int_0^1 \frac{(x_0+ph)-x_0.hdp}{(x_0+ph)-x_0}$$

$$+ \frac{y''(\S)}{2} \int_0^1 (x_0 + ph - x_0)[(x_0+ph) - (x_0+1h)].hdp$$

$$\int_{x_0}^{x_1} f(x)dx = y_0 \int_0^1 \frac{h(p-1)hdp}{-h} + y_1 \int_0^1 \frac{ph.hdp}{h} + \frac{y''(\S)}{2} \int_0^1 ph[h(p-1)]hdp$$

$$\int_{x_0}^{x_1} f(x)dx = y_0 h \left|\frac{(p-1)^2}{2}\right|_0^1 + y_1 h \left|\frac{p^2}{2}\right|_0^1 + \frac{y''(\S)}{2} h^3 \left|\frac{p^3}{3} - \frac{p^2}{2}\right|_0^1$$

$\int_{x_0}^{x_1} f(x)dx = \frac{y_0 h}{2} + \frac{y_1 h}{2} - \frac{y''(\S)}{12} h^3$ **As required.**

SIMPSON'S RULE AND ERROR TERM

Since $\quad C_k = \frac{(-1)^{n-k}.h}{k!(n-k)!} \int_0^n p(p-1)(p-2) \ldots \ldots (p-k+1)(p-k-1) \ldots \ldots (p-n)\, dp$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(i)$$

And $\quad E_n(x) = \frac{h^{n+2} y^{n+1}(\S)}{(n+1)!} \int_0^n p(p-1)(p-2) \ldots \ldots \ldots \ldots \ldots \ldots .. \ldots \ldots (p-n)dp$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (ii)$$

Putting n = 2, k = 0 in (i) we get

$$C_0 = \frac{(-1)^{2-0}.h}{0!(2-0)!}\int_0^2 (p-1)(p-2)dp = \frac{h}{2}\int_0^2 (p-1)(p-2)dp = \frac{h}{2}\int_0^2 (p^2-3p+2)dp$$

$$C_0 = \frac{h}{2}\left|\frac{p^3}{3} - \frac{3p^2}{2} + 2p\right|_0^2 = \frac{h}{2}\left(\frac{2}{3}\right) = \frac{h}{3}$$

Now Putting n = 2, k = 1 in (i) we get

$$C_1 = \frac{(-1)^1.h}{1!(2-1)!}\int_0^2 p(p-2)dp = -h\int_0^2 (p^2-2p)dp = -h\left|\frac{p^3}{3} - p^2\right|_0^2 = \frac{4}{3}h$$

Now Putting n = 2, k =2 in (i) we get

$$C_2 = \frac{(-1)^{2-2}.h}{2!(2-2)!}\int_0^2 p(p-1)dp = \frac{h}{2}\left|\frac{p^3}{3} - \frac{p^2}{2}\right|_0^2 = \frac{h}{2}\left(\frac{8}{3} - 2\right) = \frac{h}{3}$$

ERROR TERM FOR SIMPSON'S RULE

Now Putting n = 2 in (ii) we get

$$E_2(x) = \frac{h^{2+2}y^{2+1}(\S)}{(2+1)!}\int_0^2 p(p-1)(p-2)dp = \frac{h^4y^3(\S)}{3!}\int_0^2 p(p^2-3p+2)dp$$

$$E_2(x) = \frac{h^4y^3(\S)}{3!}\left|\frac{p^4}{4} - \frac{3p^3}{3} + \frac{2p^2}{2}\right|_0^2 = \frac{h^4y^3(\S)}{3!}\left(\frac{16}{4} - 8 + 4\right) = 0$$

Error term is zero so we find Global error term $E_2 = -\frac{h^3y^4(\S)}{90}$

Now for n = 3

$$\int_{x_0}^{x_3} f(x)dx = \sum_{k=0}^{3} C_k y_k = C_0y_0 + C_1y_1 + C_2y_2 + C_3y_3 + E_3(x) \quad \dots\dots\dots..(i)$$

If $C_0 = \frac{3h}{8}, C_1 = \frac{9h}{8}, C_2 = \frac{9h}{8}, C_3 = \frac{3h}{8}, \quad E_3(x) = -\frac{3h^5y^4(\S)}{80}$ **then (i) becomes**

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] - \frac{3h^5y^4(\S)}{80}$$

DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATION

It is the relation which involves the dependent variable, independent variable and Differential co-efficient i.e.

$$f(t,y)=\frac{dy}{dt}=\frac{y-y_0}{t-t_0} \qquad \Rightarrow (t-t_0)\frac{dy}{dt}=y-y_0 \qquad \Rightarrow y=y_0+(t-t_0)\frac{dy}{dt}$$

ORDINARY DIFFERENTIAL EQUATION

If differential co-efficient of Differential Equation are total, then Differential Equation is called Ordinary Differential equation. e. g. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+5y=2x$

PARTIAL DIFFERENTIAL EQUATION

If differential co-efficient of Differential Equation are partial, then Differential Equation is called Ordinary Differential equation. e. g. $\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 x}{\partial y^2}=0$

ORDER AND DEGREE OF DIFFERENTIAL EQUATION

The highest derivative involved in the equation determines the order of Differential Eq. and the power of highest derivative in Differential Eq. is called degree of D.E. for example $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y=0$ **has order "2" and degree "1"**

SOLUTION OF DIFFERENTIAL EQUATION

It is the relation which satisfies the Differential Equation as consider

$$\frac{d^2y}{dx^2}+y=0$$

Then $y=sinx, cosx, 3sinx, 20cosx$ **Are all solution of above equation.**

THE MOST GENERAL SOLUTION

It is the solution which contains as many arbitrary constants as the order of differential equation. e.g. $y''+y=0$ **Is a 2nd order Differential Eq. with constant co-efficient and general solution is** $y=c_1cosx+c_2sinx$

PARTICULAR SOLUTION

Solution which can be obtained from General Solution by giving different values to the arbitrary constants "c_1, c_2" in $y = c_1 cosx + c_2 sinx$ For example $y = 4cosx + 7sinx$

SINGULAR SOLUTION: **Solution which cannot be obtained from General Solution by giving different values to the arbitrary constants.**

SOLVE THE FOLLOWING DIFFERENCE EQUATIONS.	
$y_{k+2} - 13y_{k+1} + 36y_k = 0$	*short question*
$y_{k+2} - 7y_{k+1} + 12y_k = sin3k$	*Long question*

HOMOGENOUS DIFFERENTIAL EQUATION

A differential equation for which "$u = 0$" is a solution is called a Homogenous Differential Equation where 'u' is unknown function. In other words, a differential equation which always possesses the trivial solution "$u = 0$" is called Homogenous Differential Equation.

NON-HOMOGENOUS DIFFERENTIAL EQUATION

A differential equation for which "$u \neq 0$" (i.e. Non-Trivial solution) is a solution is called a Nonhomogeneous Differential Equation where 'u' is unknown function.

INITIAL AND BOUNDARY CONDITIONS

To evaluate arbitrary constant in the General solution we need some conditions on the unknown function or solution corresponding to some values of the independent variables. Such conditions are called Boundary or Initial conditions.

If all the conditions are given at the same value of the independent variable, then they are called Initial conditions. If the conditions are given at the end points of the independent variable, then they are called Boundary conditions.

INITIAL VALUE PROBLEM

An initial value problem for a first order Ordinary Differential Equation is the equation together with an initial condition on a specific interval $a \leq x \leq b$

Such that $y' = f(x, y)$,, $y(a) = y_a$,, $and\ x \in [a, b]$

The equation is Autonomous if (y') is independent of 'x'

BOUNDARY VALUE PROBLEM

A problem in which we solve an Ordinary Differential Equation of order two subject to condition on $y(x)$ or $y'(x)$ at two different points is called a two point boundary value problem or simply a Boundary value problem.

OR A differential equation along with one or more boundary conditions defines a boundary value problem.

CONVEX SET

A set $D \subset R^2$ is said to be convex if whenever (t_1, y_1) and (t_2, y_2) belong to 'D' then $[(1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2]$ also belong to 'D' for every "λ" in [0, 1]

LIPSCHITZ CONDITION

A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable 'y' on a set $D \subset R^2$ if a constant '$L > 0$' exists with

$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$ Whenever (t, y_1) and (t, y_2) are in 'D' and 'L' is called Lipschitz constant for 'f'

WELL – POSED PROBLEM

The initial value problem $\frac{dy}{dx} = f(x, y)$; $a \leq x \leq b$; $y(a) = a$ is said to be a well - posed problem if

A unique solution y(x) to the problem exist.

There exist constants $\in_0 > 0$ and $k > 0$ such that for any " $\in$ " with $\in_0 > \in > 0$ whenever $\delta(x)$ is continuous with $\lceil\delta(x)\rceil < \in$ $\forall\, x \in [a, b]$ and when $\delta_0 < \in$ the initial value problem $\frac{dz}{dx} = f(x, z) + \delta(x)$; $a \leq x \leq b$; $z(a) = a + \delta_0$ has a unique solution $z(x)$ that satisfies $|z(x) - y(x)| < k \in$ $\forall\, x \in [a, b]$

The problem $\frac{dz}{dx}$ is called a <u>Perturbed</u> problem associated with $\frac{dy}{dx}$

SOME STANDARD TECHNIQUES FOR SOLVING ELEMENTARY DIFFERENTIAL EQUATIONS ANALYTICALLY

❖ SECOND ORDER HOMOGENEOUS LINEAR DIFFERENCE EQUATION………………I

To solve $u_n = u_{n-1} + u_{n-2}$ $given\ that\ u_0 = 1 = u_1$ then $u_n - u_{n-1} - u_{n-2} = 0$ then zero on the right hand side signifies that is a homogeneous differential equation.

Guess $u_n = Aw^n$ then $Aw^n - Aw^{n-1} - Aw^{n-2} = 0 \Rightarrow w^2 - w - 1 = 0$
This is the auxiliary equation associated with the difference equation. Being a quadratic, the auxiliary equation signifies that the difference equation is of second order.
The two roots are readily determined $w_1 = \frac{1+\sqrt{5}}{2}$ $and\ w_2 = \frac{1-\sqrt{5}}{2}$
for any A_1substituting $A_1 w_1^n$ for u_n in $u_n - u_{n-1} - u_{n-2}$ yield zero
for any A_2substituting $A_2 w_2^n$ for u_n in $u_n - u_{n-1} - u_{n-2}$ yield zero

This suggest a general solution $u_n = A_1 w_1^n + A_2 w_2^n = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

By using initial conditions $u_0 = 1 = u_1$ one can get the values of $A_1\ and\ A_2$

That is $A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$ and $A_2 = -\frac{1-\sqrt{5}}{2\sqrt{5}}$

Then general solution becomes$u_n = \frac{1+\sqrt{5}}{2\sqrt{5}}.\left(\frac{1+\sqrt{5}}{2}\right)^n + -\frac{1-\sqrt{5}}{2\sqrt{5}}.\left(\frac{1-\sqrt{5}}{2}\right)^n$
thus $u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$ **as the final solution.**

❖ SECOND ORDER HOMOGENEOUS LINEAR DIFFERENCE EQUATION………………II

To solve $u_n = pu_{n+1} + qu_{n-1}$ $given\ that\ u_0 = 0, u_1 = 1\ and\ p + q = 1$
then $pu_{n+1} - u_n + qu_{n-1} = 0$

Guess $u_n = Aw^n$ then $pAw^{n+1} - Aw^n + qAw^{n-1} = 0 \Rightarrow pw^2 - w + q = 0$
The two roots are readily determined $w_1 = 1$ $and\ w_2 = \frac{q}{p}$
This suggest a general solution $u_n = A_1(1)^n + A_2\left(\frac{q}{p}\right)^n$ $provided\ p \neq q$
By using initial conditions $u_0 = 0, u_1 = 1\ and\ p + q = 1$one can get the values of $A_1\ and\ A_2$ That are $A_1 = -$ $A_2 = \frac{-1}{\left(\frac{q}{p}\right)^l - 1}$

thus $u_n = \frac{\left(\frac{q}{p}\right)^n - 1}{\left(\frac{q}{p}\right)^l - 1}$ **as the final solution.**

- **SECOND ORDER INHOMOGENEOUS LINEAR DIFFERENCE EQUATION**

To solve $v_n = 1 + pv_{n+1} + qv_{n-1}$ $given\ that\ v_0 = 0 = v_1 = 1\ and\ p + q = 1$
then $pv_{n+1} - v_n + qv_{n-1} = -1$
Now equation is solved in two steps. First, deem the right hand side to be zero and solve as for the homogeneous case, $v_n = A_1(1)^n + A_2\left(\frac{q}{p}\right)^n$ $\quad provided\ p \neq q$
then augmented this solution by some f(n) which has to be given further thought:
$v_n = A_1(1)^n + A_2\left(\frac{q}{p}\right)^n + f(n)$ this augmented v_n has to be such that when substituted into $pv_{n+1} - v_n + qv_{n-1}$ the result is -1
Now using $f(n) = kn$ and applying initial conditions we get the general solution
$v_n = A_1 + A_2\left(\frac{q}{p}\right)^n + \frac{n}{q-p}$

EXAMPLE: Solve the first order equation $y_{k+1} = ky_k + k^2$ given the initial condition $y_0 = 1$
SOLUTION: Values are simply found by doing indicated addition and multiplication that are $y_1 = 0, y_2 = 1, y_3 = 6, y_4 = 27, y_5 = 124$ and so on.

EXAMPLE: Solve the first order equation $y_{k+2} - 2y_{k+1} + y_k = 0$
SOLUTION: Here we have $a_1^2 = 4a_2 = 4$ the only root of $r^2 - 2r + 1 = 0$ is r=1 this means that $u_k = 1\ and\ v_k = k$ are solutions and that $y_k = c_1 + c_2k$ is a family of solutions. This is hardly surprising in view of the fact that this difference equation may be written as $\Delta^2 y_k = 0$

EXAMPLE: Solve by direct computation the second order initial value problem
$y_{k+2} = y_{k+1} + y_k \quad ; \quad y_0 = 0,\ y_1 = 1$
SOLUTION: taking $k = 0, 1, 2, 3$ ……….. We can easily find the successive values of y_k that are 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 ………………… which are known as Fibonacci numbers. The computations clearly show a growing solution but does not bring out its exact character.

METHODS FOR NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

SINGLE STEP METHODS: A series for 'y' in terms of power of 'x' form which the value of 'y' at a particular value of 'x' can be obtained by direct substitution
e.g. Taylor's, Picard's, Euler's, Modified Euler's Method.

MULTI - STEP METHODS: In multi-step methods, the solution at any point 'x' is obtained using the solution at a number of previous points.
(Predictor- corrector method, Adam's Moulton Method, Adam's Bash forth Method)

REMARK

There are some ODE that cannot be solved using the standard methods. In such situations we apply numerical methods. These methods yield the solutions in one of two forms.

- **(i) A series for 'y' in terms of powers of 'x' from which the value of 'y' can be obtained by direct substitution. e.g. Taylor's and Picard's method**
- **(ii) A set of tabulated values of 'x' and 'y'. e.g. and Euler's, Runge Kutta**

ADVANTAGE/DISADVANTAGE OF MULTI - STEP METHODS

They are not self-starting. To overcome this problem, the single step method with some order of accuracy is used to determine the starting values.

Using these methods one step method clears after the first few steps.

LIMITATION (DISADVANTAGE) OF SINGLE STEP METHODS.

For one step method it is typical, for several functions evaluation to be needed.

IMPLICIT METHODS

Method that does not directly give a formula to the new approximation. A need to get it, need an implicit formula for new approximation in term of known data. These methods also known as close methods. It is possible to get stable 3rd order implicit method.

EXPLICIT METHODS

Methods that not directly give a formula to new approximation and need an explicit formula for new approximation "y_{i+1}" in terms of known data. These are also called open methods.

Most Authorities proclaim that it is not necessary to go to a higher order method. Explain.

Because the increased accuracy is offset by additional computational effort.

If more accuracy is required, then either a smaller step size. OR an adaptive method should be used.

CONSISTENT METHOD: **A multi-step method is consistent if it has order at least one "1"**

TAYLOR'S SERIES EXPANSION

Given (x) , smooth function. Expand it at point $x = c$ then

$$f(x) = f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2!}f''(c) + \cdots \ldots \ldots \ldots$$

$\Rightarrow\ f(c) = \sum_{k=0}^{\infty} \frac{(x-c)^k}{k!} f^k$ **This is called Taylor's series of 'f' at 'c'**

If $x_0 - c = h \quad and \quad f(x) = y \quad then \Rightarrow c = x_0 + h$

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}\ y''(x_0) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots$$

MECLAURIN SERIES FROM TAYLOR'S

If we put $c = 0$ in Taylor's series then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}\ f''(0) + \frac{x^2}{3!}\ f'''(0) + \cdots \ldots . \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}\ f^k(0)$$

ADVANTAGE OF TAYLOR'S SERIES

1. **(1) One step, Explicit.**
2. **(2) Can be high order.**
3. **(3) Easy to show that global error is the same as local truncation error.**
4. **(4) Applicable to keep the error small.**

DISADVANTAGE

Need to explicit form of the derivatives of function. That is why not practical.

ERROR IN TAYLOR'S SERIES

Assume $f^k(x)$ $(0 \le k \le n)$ are continuous functions. Call

$$f_n(x) = \sum_{k=0}^{n} \frac{(x-c)^k}{k!} f^k(c) \quad \text{Then first } (n+1) \text{ term is Taylor series}$$

Then the error is

$$E_{n+1} = f(x) - f_n(x) \sum_{k=n+1}^{\infty} \frac{(x-c)^k}{k!} f^k(c) = \frac{(x-c)^{n+1}}{(n+1)!} f^{n+1} \ (\S)$$

Where '$\S'$ is some point between 'x' and 'c' .

CONVERGENCE

A Taylor's series converges rapidly if 'x' is nears 'c' and slowly (or not at all) if 'x' is for away form 'c'.

EXAMPLE

Obtain numerically the solution of $y' = t^2 + y^2$; $y(1) = 0$ using Taylor Series method to find 'y' at 1.3

SOLUTION

$y' = t^2 + y^2 \ldots \ldots \ldots .(i)$
$y'' = 2t + 2yy' \ldots \ldots \ldots .(ii) \qquad y''' = 2\left[1 + y'^2 + yy''\right] \ldots \ldots \ldots .(iii)$

$y''' = 2[yy''' + 3y'y''] \ldots \ldots \ldots .(iv)$ **...... ....and so on.**

where $y_0 = 0 \ and \ t_0 = 0 \,, \ h = t - t_0 = 0.3$

therefore $(i) \Rightarrow y_0' = 1$**,**$(ii) \Rightarrow y_0'' = 2$**,** $(iii) \Rightarrow y_0''' = 4$**,** $(iv) \Rightarrow y_0^{iv} = 12$**,.........................**

Now by using formula $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2!} y''(t_0) + \cdots \ldots \ldots \ldots \ldots \ldots$
we get

$y(1 + 0.3) = y(1.3) = 0.4132$ **as required.**

QUESTION: **Explain Taylor Series method for solving an initial value problem described by** $\frac{dy}{dx} = f(x,y);\ \dots\dots\dots\dots\dots\dots\dots (i)\ \ with\ \ y(x_0) = y_0$

SOLUTION

Here we assume that f(x,y) is sufficiently differentiable with respect to 'x' and 'y' If y(x) is exact solution of (i) we can expand by Taylor Series about the point $x = x_0$ and obtain

$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}\ y''(x_0) + \cdots \dots\dots\dots\dots\dots\dots$

Since the solution is not known, the derivatives in the above expansion are known explicitly. However 'f' is assume to be sufficiently differentiable and therefore the derivatives can be obtained directly from the given differentiable equation itself. Noting that 'f' is an implicit function of 'y' . we have $y' = f(x,y)$

$$\Rightarrow y'' = \frac{d}{dx}(y') = \frac{d}{dx}f(x,y) = \frac{df}{dx} = \frac{\partial f}{\partial x}.\frac{dx}{dx} + \frac{\partial f}{\partial y}.\frac{dy}{dx} = f_x + f_y.f$$

$\Rightarrow y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(f_x) + \frac{d}{dx}(f_y.f) \dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$

Now $\frac{d}{dx}(f_x) = \frac{\partial f_x}{\partial x}.\frac{dx}{dx} + \frac{\partial f_x}{\partial y}.\frac{dy}{dx} = f_{xx} + f_{xy}.f \quad \dots\dots\dots\dots\dots\dots\dots (a)$

$$\frac{d}{dx}(f_y.f) = f_y.\frac{df}{dx} + f.\frac{df_y}{dx} = f_y.f_x + ff_y^2 + ff_{yx} + f^2f_{yy} \quad \dots\dots\dots\dots\dots\dots\dots (b)$$

Using (a) and (b) in (ii) we get

$$\Rightarrow y''' = f_{xx} + f_{xy}.f \quad + f_y.f_x + ff_y^2 + ff_{yx} + f^2f_{yy}$$

$$\Rightarrow y''' = f_{xx} + 2ff_{xy} \quad + f_y.[f_x + ff_y] + f^2f_{yy} \qquad \therefore f_{xy} = f_{yx}$$

Continuing in this manner we can express any derivative of 'y' in term of $f(x,y)$ and its partial derivatives.

EULER'S METHOD

To find the solution of the given Differential Equation in the form of a recurrence relation $y_{m+1} = y_m + hf(t_{m,} y_m)$ Is called Euler Method

FORMULA DERIVATION

Consider the differential Equation of the first order

$$\frac{dy}{dx} = f(t, y) \qquad and \quad y(t_0) = y_0$$

Let (t_0, y_0) and (t_1, y_1) be two points of approximation curve. Then

$$y_1 - y_0 = m(x_1 - x_0) \qquad \dots\dots\dots..(i) \quad (point\ Slope\ form)$$

Given That $\frac{dy}{dt} = f(t, y) \Rightarrow \frac{dy}{dt}|_{(t_0, y_0)} = f(t_{0,} y_0) \Rightarrow m = f(t_{0,} y_0)$

$$(i) \Rightarrow y_1 - y_0 = f(t_0, y_0)(x_1 - x_0) \Rightarrow y_1 = y_0 + (x_1 - x_0)f(t_0, y_0)$$

Similarly

$$y_2 = y_1 + (x_2 - x_1)f(t_1, y_1)$$

$$y_3 = y_2 + (x_3 - x_2)f(t_2, y_2)$$

$$\vdots \qquad \vdots \qquad\qquad \vdots$$

$$y_{m+1} = y_m + (x_{m+1} - x_m)f(t_m, y_m)$$

$$\Rightarrow y_{m+1} = y_m + hf(t_m, y_m) \qquad is\ called\ Euler\ Method.$$

BASE OF EULER'S METHOD

In this method we use the property that in a small interval, a curve is nearly a Straight Line. Thus at (t_0, y_0) We approximate the Curve by a tangent at that point.

OBJECT (PURPOSE) OF METHOD

The object of Euler's Method is to obtain approximations to the well posed initial value problem $\frac{dy}{dt} = f(t, y) \quad ; a \le t \le b \ ; y(a) = a$

GEOMETRICAL INTERPRETATION

Geometrically, this method has a very simple meaning. The desired function curve is approximated by a polygon train. Where the direction of each part is determined by the value of the function $f(t, y)$ at its starting point

Also $y_{m+1} = y_m + hf(t_m, y_m)$ **Shows that the next approximation y_{m+1} is obtained at the point where the tangent to the graph of $y(t)$ at $t = t_i$ interest with the vertical line $t = t_{m+1}$**

LIMITATION OF EULER METHOD

There is too much inertia in Euler Method. One should not follow the same initial slope over the whole interval of length "h".

EULER METHOD IN VECTOR NOTATION

Consider the system $\frac{dY}{dt} = F(Y)$ where $Y = (x, y), \frac{dY}{dt} = (\frac{dx}{dt}, \frac{dy}{dt})$ and $F(Y) = (f(x, y), g(x, y))$ if we are given the initial condition $Y_0 = (x_0, y_0)$ then Euler method approximate a solution (x, y) by $(x_{k+1}, y_{k+1}) = (x_k, y_k) + \Delta t F(x_k, y_k)$

ADVANTAGE/DISADVANTAGE OF EULER METHOD

The advantage of Euler's method is that it requires only one slope evaluation and is simple to apply, especially for discretely sampled (experimental) data points. The disadvantage is that errors accumulate during successive iterations and the results are not very accurate.

EXAMPLE: **Obtain numerically the solution of** $y' = t^2 + y^2$; $y(0) = 0.5$ **using simple Euler method to find 'y' at 0.1**

SOLUTION: $y' = t^2 + y^2 = f(t, y)$ **where** $y_0 = 0.5$ and $t_0 = 0$

Then $n = \frac{t - t_0}{h} = \frac{0.1 - 0}{0.1} = 1$ $(number\ of\ steps)$ $\therefore h = t - t_0$

Now by using formula $y_{m+1} = y_m + hf(t_m, y_m)$ **we get**

$y(0.1) = y_1 = y_0 + hf(t_0, y_0) = 0.525$ **as required.**

MODIFIED EULER METHOD

Modified Euler's Method is given by the iteration formula

$$y_{m+1} == y_m + \frac{h}{2}\left[f(t_{m,}y_m) + f\left(t_{m+1,}y_{m+1}^{(1)}\right)\right]$$

Method also known as Improved Euler method sometime known as Runge Kutta method of order 2

CONVERGENCE FOR EULER METHOD

Assume that f (t, y) has a Lipschitz constant L, for the variable 'y, and that the solution y_i of the initial value problem $y' = f(t, y), t \in [a, b], y(a) = y_a$ at t_i is

Approximated by $w_i = y(t_i)$ using Euler Method

Let 'M' be an upper bound for $|y^n(t)|$ on $[a, b]$ then $[w_i - y_i] \leq \frac{Mh}{2l}\left(e^{L(t_i - a)} - 1\right)$

DARIVATION OF MODIFIED EULER METHOD

Consider the differential Equation of 1st order $\frac{dy}{dt} = f(t, y)$ and $y(t_0) = y_0$

Then by Euler's Method

$$y_1 = y_0 + hf(t_0, y_0) \qquad \because h = t_{i+1} - t_i$$

$$y_1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f\left(t_1, y_1^{(1)}\right)]$$

$$y_{2=}y_1 + \frac{h}{2}\left[f(t_{1,}y_1) + f\left(t_{2,}y_2^{(1)}\right)\right]$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$y_{m+1} = y_m + \frac{h}{2}\left[f(t_{m,}y_m) + f\left(t_{m+1,}y_{m+1}^{(1)}\right)\right]$$

EXAMPLE: Obtain numerically the solution of $y' = log(t+y)$; $y(0) = 1$ **using modified Euler method to find 'y' at 0.2**

SOLUTION: Take h = 0.1 (own choice) and $t_0 = 0$, $t_1 = t_0 + h = 0.1$, $t_2 = 0.2$

Now using Euler's method $y_1^{(1)} = y_0 + hf(t_0, y_0) = 1$

Then by using Euler's modified method

$$y_1 = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f\left(t_1, y_1^{(1)}\right)\right] = 1.002069$$

Again using Euler's method $y_2^{(1)} = y_1 + hf(t_1, y_1) = 1.006289$

Then by using Euler's modified method

$$y_2 = y_1 + \frac{h}{2}\left[f(t_1, y_1) + f\left(t_2, y_2^{(1)}\right)\right] = 1.008175 \Longrightarrow y_2 = y(0.2) \approx 1.0082$$

RUNGE KUTTA METHODS

Basic idea of Runge Kutta Methods can be explained by using Modified Euler's Method by Equation $y_{m+1} = y_n + h\ (average\ of\ slopes)$

Here we find the slope not only at 't_n' but also at several other interior points and take the weighted average of these slopes and add to 'y_n ' to get 'y_{n+1}'.

ALSO RK-Approach is to aim for the desirable features for the Taylor Series method but with the replacement of the requirement for the evaluation of the higher order derivatives with the requirement to evaluate $f(x, y)$ at some points with in the steps 'x_i' to 'x_{i+1}'

IMPORTANCE: Quite Accurate, Stable and easy to program but requires four slopes evaluation at four different points of (x,y): these slope evaluations are not possible for discretely sampled data points, because we have is what is given to us and we do not get to choose at will where to evaluate slopes. These methods do not demand prior computation of higher derivatives of $y(t)$ as in Taylor Series Method. Easy for automatic Error control. Global and local errors have same order in it.

DIFFERENCE B/W TAYLOR SERIES AND RK-METHOD

(ADVANTAGE OF RK OVER TAYLOR SERIES)

Taylor Series needs to explicit form of derivative of $f\ (t, y)$ but in RK-method this is not in demand. RK-method very extensively used.

SECOND ORDER RUNGE KUTTA METHOD

WORKING RULE: **For a given initial value problem of first order** $y' = f(x, y)$, $y(x_0) = y_0$

Suppose "$x_0, x_1, x_2 \ldots \ldots \ldots$" be equally spaced 'x' values with interval 'h'

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, ……………

Also denote $y_0 = y(x_0)$, $y_1 = y(x_1)$, $y_2 = y(x_2)$ ………………

Then for "$n = 0, 1, 2 \ldots \ldots \ldots$**" until termination do:**

$x_{n+1} = x_n + h$,, $k_n = hf(x_n, y_n)$,, $I_n = hf(x_{n+1}, y_n + k_n)$

Then $y_{n+1} = y_n + \frac{1}{2}(k_n + I_n)$ **Is the formula for second order RK-method.**

REMARK: Modified Euler Method is a special case of second order RK-Method.

IN ANOTHER WAY: **If** $k_1 = hf(x_k, y_k)$, $k_2 = hf(x_{k+1}, y_k + k_1)$

Then Equation for second order method is $y_{k+1} = y_k + \frac{1}{2}(k_1 + k_2)$
This is called Heun's Method

ANOTHER FORMULA FOR SECOND ORDER RK-METHOD

$y_{n+1} = y_n + \frac{1}{3}(2k_1 + k_2)$ **Where** $k_1 = hf(t_n, y_n)$, $k_2 = hf\left(t_n + \frac{3}{2}h,\ y_n + \frac{3}{2}k_1\right)$

LOCAL TRUNCATION ERROR IN RK-METHOD.

LTE in RK-method is the error that arises in each step simply because of the truncated Taylor series. This error is inevitable. Error of Runge Kutta method of order two involves an error of O(h^3)**.**

In General RK-method of order 'm' takes the form $x_{k+1} = x_k + w_1k_1 + w_2k_2 + \cdots + w_mk_m$

Where $k_1 = h.f(t_k, x_k)$, $k_2 = hf(t_k + a_2h, x + b_2k_1)$

$k_3 = h\,f(t_k + a_3h, x + b_3k_1 + c_3k_2)$ ………………………$k_m = h.f\left(t_k + a_mh, x + \sum_{i=1}^{m-1} \emptyset_i\, k_i\right)$

MULTI STEP METHODS OVER RK-METHOD (PREFRENCE): **Determination of** y_{i+1} **require only on evaluation of** $f\ (t, y)$ **per step. Whereas RK-method for** $n \geq 3$ **require four or more function evaluations. For this reason, multi-step methods can be twice as fast as RK-method of comparable Accuracy.**

EXAMPLE: use second order RK method to solve $\frac{dy}{dx} = \frac{y+x}{y-x} = f(x,y)\,;\, y(0) = 1$ at x =0.4 and h=0.2

SOLUTION: $\frac{dy}{dx} = \frac{y+x}{y-x} = f(x,y) \quad \dots\dots\dots\dots\dots\dots.(i)$

If 'h' is not given then use by own choice for 4 – step take h=0.1 and for 1 – step take h=0.4

Given that h=0.2 , $x_0 = 0, x_1 = x_0 + h = 0.2, x_2 = 0.4$

Now using formula of order two

$y_{n+1} = y_n + \frac{1}{3}(2k_1 + k_2)$ **Where** $k_1 = hf(x_n, y_n)$, $k_2 = hf\left(x_n + \frac{3}{2}h\,,\ y_n + \frac{3}{2}k_1\right)$

$k_1 = hf(x_0, y_0) = 0.2,\quad k_2 = hf\left(x_n + \frac{3}{2}h\,, y_n + \frac{3}{2}k_1\right) = 0.32$

$\therefore n = \frac{x - x_0}{h}$

n = 2 steps

For n = 0; $k_1 = hf(x_0, y_0) = 0.2,\quad k_2 = hf\left(x_0 + \frac{3}{2}h\,, y_0 + \frac{3}{2}k_1\right) = 0.32$

$(i) \Rightarrow y_1 = y_0 + \frac{1}{3}(2k_1 + k_2) = 1.24 \Rightarrow y(0.2) = 1.24$

For n = 1; $k_1 = hf(x_1, y_1) = 0.2769,\quad k_2 = hf\left(x_1 + \frac{3}{2}h\,, y_1 + \frac{3}{2}k_1\right) = 0.3731$

$(i) \Rightarrow y_2 = y_1 + \frac{1}{3}(2k_1 + k_2) = 1.54897 \Rightarrow y(0.4) = 1.54897$

CLASSICAL RUNGE KUTTA METHOD (RK – METHOD OF ORDER FOUR)

ALORITHM: **Given the initial value problem of first order** $y' = f(x,y)$, $y(x_0) = y_0$

Suppose "$x_0, x_1, x_2 \dots\dots\dots$" **be equally spaced 'x' values with interval 'h'**

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, …………..
Also denote $y_0 = y(x_0),\ y_1 = y(x_1),\ y_2 = y(x_2) \dots\dots\dots\dots\dots$

Then for "$n = 0, 1, 2 \dots\dots\dots..$" **until termination do:**

$x_{n+1} = x_n + h$,, $k_1 = hf(x_n, y_n)\quad k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$
$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)\quad k_4 = hf(x_n + h, y_n + k_3)$

Then $y_{n+1} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_n$

Is the formula for Runge Kutta method of order four and its error is "$O(h^5)$"

ADVANTAGE OF METHOD

- **Accurate method.**
- **Easy to compute for the use of computer.**
- **It lakes in estimating the error.**
- **Easy to program and is efficient.**

COMPUTATIONAL COMPARISON: **The main computational effort in applying the Runge Kutta method is the evaluation of 'f'. In RK – 2 the cost is two function evaluation per step. In RK – 4 require four evaluations per step.**

EXAMPLE: use 4th order RK method to solve $\frac{dy}{dx} = t + y; y(0) = 1$ from t =0 to 0.4 **taking h = 0.4**

SOLUTION: $\frac{dy}{dx} = t + y \quad \ldots\ldots\ldots\ldots\ldots\ldots.(i)$

$h = 0.1, t_0 = 0, t_1 = t_0 + h = 0.1 \ , \quad t_2 = 0.2 \ , t_3 = 0.3, t_4 = 0.4$

Now using formulas for the RK method of 4th order

$$y_{n+1} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Where $k_1 = hf(t_n, y_n)$, $k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$, $k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$ $\quad k_4 = hf(t_n + h, y_n + k_3)$

STEP I : for n=0;

$k_1 = hf(t_0, y_0) = 0.1$, $k_2 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.11$

$k_3 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1105 \quad k_4 = hf(t_0 + h, y_0 + k_3) = 0.12105$

$(ii) \Rightarrow y_1 = y(0.1) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_0 = 1.11034$

STEP II : for n=1;

$k_1 = hf(t_1, y_1) = 0.121034$, $k_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.13208$

$k_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.132638 \quad k_4 = hf(t_1 + h, y_1 + k_3) = 0.1442978$

$(ii) \Rightarrow y_1 = y(0.2) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_1 = 1.2428$

STEP III : for n=2;

$k_1 = hf(t_2, y_2) = 0.14428$, $k_2 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.156494$

$k_3 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1571047 \quad k_4 = hf(t_2 + h, y_2 + k_3) = 0.16999047$

$(ii) \Rightarrow y_1 = y(0.3) = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + y_2 = 1.399711$

THIS IS REQUIRED ANSWER

PREDICTOR - CORRECTOR METHODS

A predictor corrector method refers to the use of the predictor equation with one subsequent application of the corrector equation and the values so obtained are the final solution at the grid point.

PREDICTOR FORMULA

The explicit (open) formula used to predict approximation "y_{i+1}^{n}"is called a predictor formula.

CORRECTOR FORMULA

The implicit (closed) formula used to determine "y_{i+1}^{n}"is called Corrector Formula. This used to improve "y_{i+1}"

IN GENERAL

Explicit and Implicit formula are used as pair of formulas. The explicit formula is called 'predictor' and implicit formula is called 'corrector'

<u>Implicit methods are often used as 'corrector' and Explicit methods are used as 'predictor' in predictor-corrector method. why?</u>

Because the corresponding Local Truncation Error formula is smaller for implicit method on the other hand the implicit methods has the inherent difficulty that extra processing is necessary to evaluate implicit part.

REMARK

- **Truncation Error of predictor is** $E_p = \frac{14}{45} h^5 y_{k-1}^{(5)} \quad OR \quad \frac{28}{90} h \Delta^4 y_0'$
- **Local Truncation Error of Adam's Predictor is** $\frac{251}{720} h^5 y^{(5)}$
- **Truncation Error of Corrector is** $\frac{1}{90} h \Delta^4 y_0'$

<u>Why Should one bother using the predictor corrector method When the Single step method are of the comparable accuracy to the predictor corrector methods are of the same order?</u>

A practical answer to that relies in the actual number of functional evaluations. For example, RK - Method of order four, each step requires four evaluations where the Adams Moulton method of the same order requires only as few as two evaluations. For this reason, predictor corrector formulas are in General considerably more accurate and faster than single step methods.

REMEMBER

In predictor corrector method if values of "$y_0, y_1, y_2 \dots\dots\dots$" against the values of "$x_0, x_1, x_2 \dots\dots\dots$" are given the we use symbol predictor corrector method and in this method we use given values of"$y_0, y_1, y_2 \dots\dots\dots$"

If "$y_0, y_1, y_2 \dots\dots\dots$" Are not given against the values of "$x_0, x_1, x_2 \dots\dots\dots$" then we first find values of "$y_0, y_1, y_2 \dots\dots\dots$" by using RK - method
OR By using formula $\forall\ j = 1, 2, 3 \dots\dots\dots n$

$$y_j = y_0 + (jh)y_0' + \frac{(jh)^2}{2!} y_0'' + \frac{(jh)^3}{3!} y_0''' + \dots\dots\dots\dots$$

BASE (MAIN IDEA) OF PREDICTOR CORRECTOR METHOS

In predictor corrector methods a predictor formula is used to predict the value of 'y' at t_{n+1} and then a corrector formula is used to improve the value of y_{n+1}
Following are predictor – corrector methods

1. Milne's Method
2. Adam – Moulton method

MILNE'S METHOD

It's a multi-step method. In General, Milne's Predictor – Corrector pair can be written as

P: $y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) \quad n \geq 3$

C: $y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) \quad n \geq 3$

REMARK: Magnitude of truncation error in Milne's corrector formula is $\frac{1}{90} h\, \Delta^4 y_0'$ and truncation error in Milne's predictor formula is $\frac{28}{90} h\, \Delta^4 y_0'$
stable, convergent, efficient, accurate, compeer friendly.

ALGORITHM

- First predict the value of y_{n+1} by above predictor formula.
  Where derivatives are computed using the given differential equation itself.
- Using the predicted value "y_{n+1}" we calculate the derivative $y'_{n+1}$from the given differential Equation.
- Then use the corrector formula given above for corrected value of y_{n+1} . Repeat this process.

EXAMPLE: use Milne's method to solve $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$ and compute y(0.8)

SOLUTION: $h = 0.2, x_0 = 0, x_1 = x_0 + h = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$ *also* $y_0 = 0$

Now by using Euler's method $\Longrightarrow y_{m+1} = y_m + hf(t_m, y_m)$

$for\ m = 0; \quad \Longrightarrow y_1 = y_0 + hf(t_0, y_0) = 0.2 = y(0.2)$

$for\ m = 1; \quad \Longrightarrow y_2 = y_1 + hf(t_1, y_1) = 0.48 = y(0.4)$

$for\ m = 2; \quad \Longrightarrow y_3 = y_2 + hf(t_2, y_2) = 0.73 = y(0.6)$

Now $y'_n = 1 + y_n^2$

For n=1 $\Longrightarrow y'_1 = 1 + y_1^2 = 1.04$

For n=2 $\Longrightarrow y'_2 = 1 + y_2^2 = 1.16$

For n=3 $\Longrightarrow y'_3 = 1 + y_3^2 = 1.36$

Now using Milne's Predictor formula

P: $y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) \quad n \geq 3$

$y_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) = 0.98 \Longrightarrow y'_4 = 1 + y_4^2 = 1.9604$

Now using corrector formula

C: $y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) \qquad n \geq 3$

$y_4 = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) = 1.05 = y(0.8)$