Every open cover has a finite sub.

CHAPTER 6

COMPACTNESS

6.0 INTRODUCTION

We have seen that the concept of completeness is the abstraction of a property of the real number system. The concept of compactness is also an abstraction of an important property possessed by subsets of **R** which are closed and bounded. This property is known as Heine Borel theorem which states that if $I \subseteq \mathbf{R}$ is a closed interval, any family of open intervals in **R** whose union contains I has a finite subfamily whose union contains I. We now introduce the class of *compact metric spaces* in which the conclusion of Heine Borel theorem is valid.

6.1 COMPACT METRIC SPACES

Definition. Let M be a metric space. A family of open sets $\{G_\alpha\}$ in M is called an **open cover** for M if $\cup G_\alpha = M$.

A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a **subcover.**

A metric space M is said to be **compact** if every open cover for M has finite subcover.

(i.e) for each family of open sets $\{G_\alpha\}$ such that $\cup G_\alpha = M$, there exist a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} = M$.

connected ope, closed separable complete

Example 1. **R** with usual metric is not compact.

Proof. Consider the family of open intervals $\{(-n, n)/n \in \mathbf{N}\}$.

This is a family of open sets in **R**.

Clearly $\bigcup_{n=1}^{\infty}(-n, n) = \mathbf{R}$.

$\therefore$ $\{(-n, n)/n \in \mathbf{N}\}$ is an open cover for $\mathbf{R}$ and this open cover has no finite subcover.

$\therefore$ $\mathbf{R}$ is not compact.

Example 2. $(0, 1)$ with usual metric is not compact.

Proof. Consider the family of open intervals $\left\{\left(\frac{1}{n}, 1\right)/n = 2, 3, ...\right\}$.

Clearly $\bigcup_{n=2}^{\infty}\left(\frac{1}{n}, 1\right) = (0, 1)$.

$\therefore$ $\left\{\left(\frac{1}{n}, 1\right)/n = 2, 3, ...\right\}$ is an open cover for $(0, 1)$ and this open cover has no finite subcover.

Hence $(0, 1)$ is not compact.

Example 3. $[0, \infty)$ with usual metric is not compact.

Proof. Consider the family of intervals $\{[0, n)/n \in \mathbf{N}\}$.

$[0, n)$ is open in $[0, \infty)$ for each $n \in \mathbf{N}$.

[0, n]

Also $\bigcup_{n=1}^{\infty}[0, n) = [0, \infty)$.

$\therefore$ $\{[0, n)/n \in \mathbf{N}\}$ is an open cover for $[0, \infty)$ and this open cover has no finite subcover.

Hence $[0, \infty)$ is not compact.

Example 4. Let M be an infinite set with discrete metric. Then M is not compact.

Proof. Let $x \in M$. Since M is a discrete metric space $\{x\}$ is open in M.

Also $\bigcup_{x \in M}\{x\} = M$.

Hence $\{\{x\}/x \in M\}$ is an open cover for M and since M is infinite, this open cover has no finite subcover.

Hence M is not compact.

Example 5. We will prove in 6.2 that any closed interval $[a, b]$ with usual metric is compact.

Theorem 6.1 Let M be a metric space. Let $A \subseteq M$. A is compact iff given a family of open sets $\{G_\alpha\}$ in M such that $\cup G_\alpha \supseteq A$ there exists a subfamily $G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Proof. Let A be a compact subset of M.

Let $\{G_\alpha\}$ be a family of open sets in M such that $\cup G_\alpha \supseteq A$.

Then $(\cup G_\alpha) \cap A = A$.

$\therefore \cup (G_\alpha \cap A) = A$.

Also $G_\alpha \cap A$ is open in A. (refer theorem 2.6)

$\therefore$ The family $\{G_\alpha \cap A\}$ is an open cover for A.

Since A is compact this open cover has a finite subcover, say, $G_{\alpha_1} \cap A, G_{\alpha_2} \cap A, \ldots\ldots, G_{\alpha_n} \cap A$.

$$\therefore \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A.$$

$$\therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap A = A.$$

$$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$$

Conversely let $\{H_\alpha\}$ be an open cover for A.

$\therefore$ Each H_α is open in A.

$\therefore H_\alpha = G_\alpha \cap A$ where G_α is open in M.

Now, $\cup H_\alpha = A$.

$\therefore \cup (G_\alpha \cap A) = A$.

$\therefore (\cup G_\alpha) \cap A = A$

$\therefore \cup G_\alpha \supseteq A$.

Hence by hypothesis there exists a finite subfamily

$$G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n} \text{ such that } \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$$

$$\therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i} \right) \cap A = A.$$

$$\therefore \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A.$$

$$\therefore \bigcup_{i=1}^{n} H_{\alpha_i} = A.$$

Thus $\{H_{\alpha_1}, H_{\alpha_2}, \ldots\ldots, H_{\alpha_n}\}$ is a finite subover of the open cover $\{H_\alpha\}$

$\therefore A$ is compact.

Theorem 6.2 Any compact subset A of a mectric space M is bounded.

Proof. Let $x_o \in A$.

Consider $\{B(x_o, n) / n \in \mathbb{N}\}$.

$$\text{Clearly } \bigcup_{n=1}^{\infty} B(x_o, n) = M.$$

$$\therefore \bigcup_{n=1}^{\infty} B(x_o, n) \supseteq A.$$

Since A is compact there exists a finite subfamily say,

$B(x_o, n_1)$, $B(x_o, n_2)$, ..., $B(x_o, n_k)$ such that $\bigcup_{i=1}^{k} B(x_0, n_1) \supseteq A$.

Let $n_o = \max\{n_1, n_2, \ldots, n_k\}$.

Then $\bigcup_{i=1}^{k} B(x_o, n_i) = B(x_o, n_0)$.

$\therefore\ B(x_o, n_o) \supseteq A$.

We know that $B(x_o, n_o)$ is a bounded set and a subset of a bounded set is bounded. Hence A is bounded.

Note. The converse of the above theorem is not true.

For example, $(0, 1)$ is a bounded subset of **R**.

But it is not compact. (refer example 2 of 6.1)

Theorem 6.3 Any compact subset A of a metric space (M, d) is closed.

Proof. To prove that A is closed we shall prove that A^c is open.

Let $y \in A^c$ and let $x \in A$. Then $x \neq y$.

$\therefore\ d(x, y) = r_x > 0$.

It can be easily verified that $B(x, \frac{1}{2} r_x) \cap B(y, \frac{1}{2} r_x) = \Phi$.

Now consider the collection $\{B(x, \frac{1}{2} r_x) / x \in A\}$.

Clearly $\bigcup_{x \in A} B(x, \frac{1}{2} r_x) \supseteq A$.

Since A is compact there exists a finite number of such open balls say,

$$B(x_1, \tfrac{1}{2} r_{x_1}), \ldots, B(x_n, \tfrac{1}{2} r_{x_n}) \text{ such that } \bigcup_{i=1}^{n} B(x_i, \tfrac{1}{2} r_{x_i}) \supseteq A. \qquad \ldots\ldots (1)$$

A^c

Now, let $V_y = \bigcap_{i=1}^{n} B(y, \frac{1}{2}r_{x_i})$.

Clearly V_y is an open set containing y.

Since $B(y, \frac{1}{2}r_y) \cap B(x, \frac{1}{2}r_x) = \Phi$, we have $V_y \cap B(x_i, \frac{1}{2}r_{x_i}) = \Phi$ for each $i = 1, 2, , \ldots, n$.

$$\therefore\ V_y \cap \left[\bigcup_{i=1}^{n} B(x_i, \tfrac{1}{2}r_{x_i}) \right] = \Phi.$$

$$\therefore\ V_y \cap A = \Phi \quad [\text{ by } (1)\ ].$$

$$\therefore\ V_y \subseteq A^c.$$

$$\therefore\ \bigcup_{y \in A^c} V_y = A^c \text{ and each } V_y \text{ is open.}$$

$\therefore$ A^c is open. Hence A is closed.

Note 1. The converse of the above theorem is not true.

For example, $[0, \infty)$ is a closed subset of **R**. But it is not compact (refer eg. 3 of 6.1)

Note 2. It follows from theorems 6.2 and 6.3 that any compact subset of a metric space is closed and bounded.

Theorem 6.4 A closed subspace of a compact metric space is compact.

Proof. Let M be a compact metric space. Let A be a non-empty closed subset of M.

We claim that A is compact. every open cover has a finite s-c

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in M such that

$$\bigcup_{\alpha \in I} G_\alpha \supseteq A.$$

open cover

A^c is open

$\therefore\ A^c \cup \left[\bigcup_{\alpha\in I} G_\alpha\right] = M.$

Also A^c is open. (since A is closed).

$\therefore\ \{G_\alpha/\alpha \in I\} \cup \{A^c\}$ is an open cover for M.

Since M is compact it has a finite subcover say

$$G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}, A^c.$$

$\cup$ sub cover $\Rightarrow M$

$$\therefore\ \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cup A^c = M.$$

$$\therefore\ \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$$

$\therefore\ A$ is compact.

Exercises

1. Give an example of an open cover which has no finite subcover for the following subsets of **R**.

(i) (5, 6) (ii) (5,∞) (iii) [5, ∞) (iv) [7, 9)

2. Show that every finite metric space is compact.

3. Give an example of a connected subset of **R** which is not compact. (Hint. Any interval in **R** is connected) (0,1)

4. A and B are two compact subsets of a metric space M. Prove that $A \cup B$ is also compact.

6.2 COMPACT SUBSETS OF R.

We have already proved that every compact subset of a metric space is closed and bounded.

$x \to$ non empty

$x_1 \to$ cover

$x_n \to c = b$

However the converse is not true.

For example, consider an infinite discrete metric space (M, d).

Let A be an infinite subset of M.

Then A is bounded since $d(x, y) \leq 1$ for all $x, y \in A$.

Also A is closed since any subset of a discrete metric space is closed.

Hence A is closed and bounded.

However A is not compact (refer example 4 of 6.1)

In this section we shall prove that for **R** with usual metric the converse is also true.

Theorem 6.5 (Heine Borel theorem) ★ 8 m Imp

U.Q *Any closed interval $[a, b]$ is a compact subset of* **R**.

Proof. Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in **R** such that $\bigcup_{\alpha \in I} G_\alpha \supseteq [a, b]$. claim

set a cover pannum $[a,b] \to$ comp

$(a \leq x \leq b)$

Let $S = \{x / x \in [a, b]$ and $[a, x]$ can be covered by a finite number of G_α's.

Clearly $a \in S$ and hence $S \neq \Phi$. ① 10

Also S is bounded above by b. u.b $= b$ all G_α cover in $[a, b]$

Let c denote the $l.u.b.$ of S. finite G_α cover in [

Clearly $c \in [a, b]$. $\subseteq \bigcup G_\alpha$

$\therefore c \in G_{\alpha_1}$ for some $\alpha_1 \in I$.

any one G_α

Since G_{α_1} is open, there exists $\varepsilon > 0$ such that

$B(c, \varepsilon) \subseteq G_{\alpha_1}$

$(c - \varepsilon, c + \varepsilon)$

$$(c - \varepsilon, c + \varepsilon) \subseteq G_{\alpha_1}.$$ ②

Choose $x_1 \in [a, b]$ such that $x_1 < c$ and $[x_1, c] \subseteq G_{\alpha_1}$. u.b

$x_1 < c$, $c \in G_{\alpha_1}$

Now, since $x_1 < c$, $[a, x_1]$ can be covered by a finite number of G_α's.

$[x_1, c], [a, x_1]$

These finite number of G_α's together with G_{α_1} covers $[a, c]$.

$\therefore$ By definition of S, $\boxed{c \in S.}$

Now, we claim that $c = b$. (3)

c is upper bound

Suppose $c \neq b$.

Then choose $x_2 \in [a, b]$ such that $x_2 > c$ and $[c, x_2] \subseteq G_{\alpha_1}$.

$[a, x_2]$

As before, $[a, x_2]$ can be covered by a finite number of G_α's.

Hence $x_2 \in S$.

But $x_2 > c$ which is a contradiction, since c is the $l.\,u.\,b.$ of S.

$\therefore$ $c = b$.

$\therefore$ $[a, b]$ can be covered by a finite number of G_α's.

$\therefore$ $[a, b]$ is a compact subset of **R**.

Theorem 6.6 A subset A of **R** is compact iff A is closed and bounded.

Proof. If A is compact then A is closed and bounded.

Conversely, let A be subset of **R** which is closed and bounded.

Since A is bounded we can find a closed interval $[a, b]$ such that $A \subseteq [a, b]$.

Since A is closed in **R**, A is closed in $[a, b]$ also.

Thus A is a closed subset of the compact space $[a, b]$.

Hence A is compact. (by theorem 6.4)

Exercises

1. Determine which of the following subset of **R** are comapct.

(i) **Z** (ii) **Q** (iii) $[1, 2]$

(iv) $(3, 4)$ (v) $(0, \infty)$ (vi) $[1, 2] \cup [3, 4]$

compact

(vii) $[1, 3] \cap [3, 4]$ (viii) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$

, 3, 4] (ix) $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$.

Saran 2. If A and B are compact subsets of $\mathbf{R}$ prove that $A \cap B$ is also a compact subset of $\mathbf{R}$.

$A \cup B$

6.3 EQUIVALENT CHARACTERISATIONS FOR COMPACTNESS

In this section we obtain several equivalent characterisation for compactness in a metric space.

script $\mathfrak{I}$.

Definition. A family $\mathfrak{I}$ of subsets of a set M is said to have the **finite intersection property** if any finite members of $\mathfrak{I}$ have non-empty intersection.

Example. In $\mathbf{R}$ the family of closed intervals $\mathfrak{I} = \{[-n, n] / n \in \mathbf{N}\}$ has finite intersection property.

Theorem 6.7 *A metric space M is compact iff any family of closed sets with finite intersection property has non-empty intersection.*

2

Proof. Suppose M is compact.

Let $\{A_\alpha\}$ be a family of closed subsets of M with finite intersection property.

We claim that $\cap A_\alpha \neq \Phi$.

Suppose $\cap A_\alpha = \Phi$ then $(\cap A_\alpha)^c = \Phi^c$.

$\therefore \cup A_\alpha^c = M$.

Also, since each A_α is closed, A_α^c is open.

$\therefore \{A_\alpha^c\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say,

$A_1{}^c, A_2{}^c, \ldots\ldots, A_n{}^c$.

$$\therefore \bigcup_{i=1}^{n} A_i^c = M.$$

$$\therefore \left(\bigcap_{i=1}^{n} A_i\right)^c = M.$$

$\therefore \bigcap_{i=1}^{n} A_i = \Phi$ which is a contradiction to the finite intersection property.

$$\therefore \cap A_\alpha \neq \Phi.$$

Conversely, suppose that each family of closed sets in M with finite intersection property has non empty intersection.

To prove that M is compact, let $\{G_\alpha / \alpha \in I\}$ be an open cover for M.

$$\therefore \bigcup_{\alpha \in I} G_\alpha = M.$$

$$\therefore \left(\bigcup_{\alpha \in I} G_\alpha\right)^c = M^c.$$

$$\therefore \bigcap_{\alpha \in I} G_\alpha^c = \Phi.$$

Since G_α is open, G_α^c is closed for each α.

$\therefore \mathfrak{I} = \{G_\alpha^c / \alpha \in I\}$ is a family of closed sets whose intersection is empty.

Hence by hypothesis this family of closed sets does not have the finite intersection property.

Hence there exists a finite sub-collection of $\mathfrak{I}$ say,

$\{G_1^c, G_2^c, \ldots\ldots G_n^c\}$ such that $\bigcap_{i=1}^{n} G_i^c = \Phi$.

$$\therefore \left(\bigcup_{i=1}^{n} G_i\right)^c = \Phi.$$

$$\therefore \bigcup_{i=1}^{n} G_i = M.$$

$\therefore \{G_1, G_2, \ldots\ldots\ldots, G_n\}$ is a finite subcover of the given open cover.

Hence M is compact.

Definition. A metric space M is said to be **totally bounded** if for every $\varepsilon > 0$ there exists a finite number of elements $x_1, x_2, \ldots\ldots, x_n \in M$ such that $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup B(x_n, \varepsilon) = M$.

A non-empty subset A of a metric space M is said to be totally bounded if the subspace A is a totally bounded metric space.

Theorem 6.8 Any compact metric space is totally bounded.

Proof. Let M be a compact metric space.

Then $\{B(x, \varepsilon)/x \in M\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say, $B(x_1, \varepsilon), B(x_2, \varepsilon), \ldots\ldots, B(x_n, \varepsilon)$.

$\therefore M = B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup (B(x_n, \varepsilon)$.

$\therefore M$ is totally bounded.

Theorem 6.9 Let A be a subset of a metric space M. If A is totally bounded then A is bounded.

Proof. Let A be a totally bounded subset of M. Let $\varepsilon > 0$ be given.

Then there exists a finite number of points $x_1, x_2, \ldots\ldots, x_n \in A$, such that $B_1(x_1, \varepsilon) \cup B_1(x_2, \varepsilon) \cup \ldots \cup B_1(x_n, \varepsilon) = A$, where $B_1(x_i, \varepsilon)$ is an open ball in A.

Further we know that an open ball is a bounded set.

Thus A is the union of a finite number of bounded sets and hence A is bounded.

Note. The converse of the above theorem is not true.

For, let M be an infinite set with discrete metric.

Clearly M is bounded.

Now, $B(x, \frac{1}{2}) = \{x\}$.

Since M is infinite, M cannot be written as the union of a finite number of open balls $B(x, \frac{1}{2})$.

$\therefore$ M is not totally bounded.

Definition. Let (x_n) be sequence in a metric space M.

Let $n_1 < n_2 < \ldots\ldots < n_k < \ldots\ldots.$ be an increasing sequence of positive integers. Then (x_{n_k}) is called a **subsequence** of (x_n).

Theorem 6.10 A metric space (M, d) is totally bounded iff every sequence in M has a Cauchy subsequence.

Proof. Suppose every sequence in M has a Cauchy subsequence.

We claim that M is totally bounded.

Let $\varepsilon > 0$ be given. Choose $x_1 \in M$.

If $B(x_1, \varepsilon) = M$ then obviously M is totally bounded.

If $B(x_1, \varepsilon) \neq M$, choose $x_2 \in M - B(x_1, \varepsilon)$ so that $d(x_1, x_2) \geq \varepsilon$.

Now, if $B(x_1, \varepsilon) \cup (Bx_2, \varepsilon) = M$ the proof is complete.

If not choose $x_3 = M - [B(x_1, \varepsilon) \cup B(x_2, \varepsilon)]$ and so on.

Suppose this process does not stop at a finite stage.

Then we obtain a sequence $x_1, x_2, \ldots\ldots, x_n, \ldots\ldots.$ such that $d(x_n, x_m) \geq \varepsilon$ if $n \neq m$.

Clearly this sequence (x_n) cannot have a Cauchy subsequence which is a contradiction.

Hence the above process stops at a finite stage and we get a finite set of points $\{x_1, x_2, \ldots\ldots, x_n\}$ such that

$$M = B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup B(x_n, \varepsilon).$$

$\therefore$ M is totally bounded.

Conversely suppose M is totally bounded.

Let $S_1 = \{x_{i_1}, x_{i_2}, \ldots\ldots, x_{i_n} \ldots\ldots\}$ be a sequence in M.

If one term of the sequence is infinitely repeated then S_1 contains a constant subsequence which is obviously a Cauchy subsequence.

Hence we assume that no term of S_1 is infinitely repeated so that the range of S is infinite.

Now, since M is totally bounded M can be covered by a finite number of open balls of radius $\frac{1}{2}$.

Hence atleast one of these balls must contain an infinite number of terms of the sequence S_1.

$\therefore$ S_1 contains a subsequence $S_2 = (x_{2_1}, x_{2_2}, \ldots\ldots, x_{2_n}, \ldots\ldots)$ all terms of which lie within an open ball of radius $\frac{1}{2}$.

Similarly S_2 contains a sub sequence $S_3 = (x_{3_1}, \ldots\ldots, x_{3_n}, \ldots\ldots)$ all terms of which lie within an open ball of radius $\frac{1}{3}$.

We repeat this process of forming successive subsequences and finally we take the diagonal sequence.

$S = (x_{1_1}, x_{2_2}, \ldots\ldots x_{n_n} \ldots\ldots)$.

We claim that S is a Cauchy subsequence of S_1.

If $m > n$ both x_{m_m} and x_{n_n} lie within an open ball of radius $\frac{1}{n}$.

$\therefore\ d(x_{m_m}, x_{n_n}) < \frac{2}{n}$.

Hence $d(x_{m_m}, x_{n_n}) < \varepsilon$ if $n, m > \frac{2}{\varepsilon}$.

This shows that S is a Cauchy subsequence of S_1.

Thus every sequence in M contains a Cauchy subsequence.

Corollary. A non-empty subset of a totally bounded set is totally bounded.

Proof. Let A be a totally bounded subset of a metric space M.

Let B be a non-empty subset of A.

Let (x_n) be a sequence in B.

$\therefore$ (x_n) is a sequence in A.

Since A is totally bounded (x_n) has a Cauchy subsequence. (Thrm : 6.10)

Thus every sequence in B has a Cauchy subsequence.

$\therefore$ B is totally bounded. all seq are cauchy sub

Definition. A metric space M is said to be **sequentially compact** if every sequence in M has a convergent sub-sequence.

Theorem 6.11. Let (x_n) be a Cauchy sequence in a metric space M. If (x_n) has a subsequence (x_{n_k}) converging to x, then (x_n) converges to x.

Gn: cauchy seq def

Proof. Let $\varepsilon > 0$ be given. Since (x_n) is a Cauchy sequence, there exists a positive integer m_1 such that $d(x_n, x_m) < \frac{1}{2}\varepsilon$ for all $n, m \geq m_1$ ..... (1)

Also, since $(x_{n_k}) \to x$, there exists a positive integer m_2 such that

$(x_{n_k}) \to x$ $d(x_{n_k}, x) < \frac{1}{2}\varepsilon$ for all $n_k \geq m_2$ ..... (2)

subseq

Let $m_o = \max\{m_1, m_2\}$ and fix $n_k \geq m_o$ desired

reason

Then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$ $d(x_n, x) < \varepsilon$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ for all } n \geq m_o \qquad [\text{ by (1) and (2) }]$$

$$= \varepsilon \text{ for all } n \geq m_o.$$

Hence $(x_n) \to x$..

8m ★ Imp

Theorem 6.12 In a metric space M the following are equivalent.

(i) M is compact.

(ii) Any infinite subset of M has a limit point.

(iii) M is sequentially compact.

(iv) M is totally bounded and complete.

Proof. (i) $\Rightarrow$ (ii)

Let A be an infinite subset of M.

Suppose A has no limit point in M.

Let $x \in M$.

A has no limit pt

Since x is not a limit point of A there exists an open ball $B(x, r_x)$ such that $B(x, r_x) \cap (A - \{x\}) = \Phi$.

infinite sub of M B

$$\therefore B(x, r_x) \cap A = \begin{cases} \{x\} \text{ if } x \in A \\ \Phi \text{ if } x \notin A \end{cases}$$

since M is compact

Now, $\{B(x, r_x) / x \in M\}$ is open cover for M.

Also each $B(x, r_x)$ covers atmost one point of the infinite set A.

Hence this open cover cannot have a finite sub cover which is a contradiction to (i). Hence A has atleast one limit point.

(ii) ⇒ (iii).

Let (x_n) be a sequence in M.

every constant

(If one term of the sequence is infinitely repeated then (x_n) contains a constant subsequence which is convergent.)

Otherwise (x_n) has an infinite number of terms. (ii)

By hypothesis this infinite set has a limit point, say x.

By theorem 2.15 for any $r > 0$ the open ball $B(x, r)$ contains infinite number of terms of the sequence (x_n). (Prob 6(a))

Now, choose a positive integer n_1 such that $x_{n_1} \in B(x, 1)$.

Then choose $n_2 > n_1$ such that $x_{n_2} \in B(x, \frac{1}{2})$.

In general for each positive integer k choose n_k such that $n_k > n_{k-1}$ and $x_{n_k} \in B(x, \frac{1}{k})$.

Clearly (x_{n_k}) is a subsequence of (x_n).

Also $d(x_{n_k}, x) < \frac{1}{k}$.

$\therefore$ $(x_{n_k}) \to x$.

Thus (x_{n_k}) is a convergent subsequence of (x_n).

Hence M is sequentially compact.

(iii) ⇒ (iv).

By hypothesis every sequence in M has a convergent subsequence. But every convergent sequence is a Cauchy sequence.

Thus every sequence in M has a Cauchy subsequence.

$\therefore$ By theorem 6.10, M is totally bounded.

Now we prove that M is complete.

Let (x_n) be a Cauchy sequence in M.

By hypothesis (x_n) contains a convergent subsequence (x_{n_k}).

Let $(x_{n_k}) \to x$. (say)

Then by theorem 6.11, $(x_n) \to x$.

$\therefore$ M is complete.

(iv) $\Rightarrow$ (i)

Suppose M is not compact.

Then there exists an open cover $\{G_\alpha\}$ for M which has no finite subcover.

Let $\boxed{r_n = \frac{1}{2^n}}$.

(Since, M is totally bounced, M can be covered by a finite number of open balls of radius r_1.) $r = 1/2$

M is not comp Since M cannot be covered by a finite number of G_α 's atleast one of these open balls, say $B(x_1, r_1)$ cannot be covered by a finite number of G_α 's

Now, $B(x_1, r_1)$ is totally bounded.

Hence as before we can find $x_2 \in B(x_1, r_1)$ such that $B(x_2, r_2)$ cannot be covered by a finite number of G_α 's. $r = 1/4$

Proceeding like this we obtain a sequence (x_n) in M such that $B(x_n, r_n)$ cannot be covered by a finite number of G_α 's and

$$x_{n+1} \in B(x_n, r_n) \text{ for all } n.$$

Now, $d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots + d(x_{n+p-1}, x_{n+p})$

$$< r_n + r_{n+1} + \ldots\ldots + r_{n+p-1}$$

$$= \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots\cdots\cdots + \frac{1}{2^{n+p-1}}$$

$$= \frac{1}{2^{n-1}} \left(\frac{1}{2} + \frac{1}{2^2} + \cdots\cdots + \frac{1}{2^p} \right)$$

$$< \frac{1}{2^{n-1}}.$$

$\therefore$ (x_n) is a Cauchy sequence in M.

Since M is complete there exists $x \in M$ such that $(x_n) \to x$.

Now, $x \in G_\alpha$ for some α.

open def n:

Since G_α is open we can find $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq G_\alpha$. ..... (1)

We have $(x_n) \to x$ and $(r_n) = \left(\frac{1}{2^n}\right) \to 0$.

Hence we can find a positive integer n_1 such that $d(x_n, x) < \frac{1}{2}\varepsilon$ and $r_n < \frac{1}{2}\varepsilon$ for all $n \geq n_1$.

Now, fix $n \geq n_1$. Small

We claim that $B(x_n, r_n) \subseteq B(x, \varepsilon)$.

Let $y \in B(x_n, r_n)$

$\therefore$ $d(y, x_n) < r_n < \frac{1}{2}\varepsilon$ (since $n \geq n_1$)

Now, $d(y, x) \leq d(y, x_n) + d(x_n, x)$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

$\therefore$ $y \in B(x, \varepsilon)$.

$\therefore$ $B(x_n, r_n) \subseteq B(x, \varepsilon) \subseteq G_\alpha$ (by (1))

Thus $B(x_n, r_n)$ is covered by the single set G_α which is a contradiction since $B(x_n, r_n)$ cannot be covered by a finite number of G_α's.

Hence M is compact.

Theorem 6.13. **R** with usual metric is complete.

Proof. Let (x_n) be a Cauchy sequence in **R**. interval Thrm

Then (x_n) is a bounded sequence and hence is contained in a closed interval $[a, b]$.

result

Now, $[a, b]$ is compact and hence is complete.

Hence (x_n) converges to some point $x \in [a, b]$.

Thus every Cauchy sequence (x_n) in **R** converges to some point x in **R** and hence **R** is complete.

Solved problems

Problem 1. Give an example of a closed and bounded subset of l_2 which is not compact.

Solution. Consider $\mathbf{0} = (0, 0, 0, \ldots\ldots) \in l_2$.

Consider the closed ball $B[\mathbf{0}, 1]$.

Clearly, $B[\mathbf{0}, 1]$ is bounded.

Also, $B[\mathbf{0}, 1]$ is a closed set. (since clos ball)

We claim that $B[\mathbf{0}, 1]$ is not compact.

Consider $e_1 = (1, 0, 0, \ldots)$; $e_2 = (0, 1, 0, \ldots)$; ......

$$\ldots\ldots\ldots e_n = (0, 0, 0, \ldots, 1, 0, \ldots).$$

Now, $d(\mathbf{0}, e_n) = 1$ and hence $e_n \in B[0, 1]$ for all n.

Thus (e_n) is a sequence in $B[\mathbf{0}, 1]$.

d(en, em) = √2 if n ≠ m
0 if n = m

Also $d(e_n, e_m) = \sqrt{2}$ if $n \neq m$.

Hence the sequence (e_n) does not contain a Cauchy subsequence.

$\therefore$ $B[0, 1]$ is not totally bounded.

$\therefore$ $B[0, 1]$ is not compact.

M is totally bounded ⇒ M is compact

Problem 2. Prove that any totally bounded metric space is separable.

Solution. Let M be a totally bounded metric space.

For each natural number n let $A_n = \{x_{n_1}, x_{n_2}, \ldots, x_{n_k}\}$ be a subset of M such that $\bigcup_{i=1}^{k} B\left(x_{n_i}, \frac{1}{n}\right) = M$. ..... (1)

Let $A = \bigcup_{n=1}^{\infty} A_n$.

Since each A_n is finite, A is a countable subset of M.

We claim that A is dense in M.

Let $B(x, \varepsilon)$ be any open ball.

Choose a natural number n such that $\frac{1}{n} < \varepsilon$.

Now, $x \in B\left(x_{n_i}, \frac{1}{n}\right)$ for some i. (by (1)).

$\therefore$ $d(x_{n_i}, x) < \frac{1}{n} < \varepsilon$.

$\therefore$ $(x_{n_i}) \in B(x, \varepsilon)$.

$\therefore$ $B(x, \varepsilon) \cap A \neq \Phi$.

Thus every open ball in M has non-empty intersection with A.

Hence by theorem 2.17, A is dense in M.

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Thus A is a countable dense subset of M.

Hence M is separable.

A intersects every open ball

Problem 3. Prove that any bounded sequence in $\mathbf{R}$ has a convergent subsequence.

Solution. Let (x_n) be a bounded sequence in $\mathbf{R}$.

Then there exists a closed interval $[a, b]$ such that $x_n \in [a, b]$ for all n.

Thus (x_n) is a sequence in the compact metric space $[a, b]$.

Hence by theorem 6.12, (x_n) has a convergent sub-sequence.

U.Q.

Problem 4. Prove that the closure of a totally bounded set is totally bounded.

Solution. Let A be a totally bounded subset of a metric space M.

We claim that $\overline{A}$ is totally bounded.

We shall show that every sequence in $\overline{A}$ contains a Cauchy subsequence.

Let (x_n) be a sequence in $\overline{A}$.

Let $\varepsilon > 0$ be given.

Then since $x_n \in \overline{A}$, $B(x_n, \frac{1}{3}\varepsilon) \cap A \neq \Phi$.

Choose $y_n \in B(x_n, \frac{1}{3}\varepsilon) \cap A$.

$\therefore\ d(y_n, x_n) < \frac{1}{3}\varepsilon.$ ..... (1)

Now, (y_n) is a sequence in A. Since A is totally bounded (y_n) contains a Cauchy sequence say (y_{n_k}).

Hence there exists a natural number m such that

Sub Aeq

$$d(y_{n_i}, y_{n_j}) < \frac{1}{3}\varepsilon \text{ for all } n_i, n_j \geq m \qquad \text{..... (2)}$$

$$\therefore\ d(x_{n_i}, x_{n_j}) \leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, y_{n_j}) + d(y_{n_j}, x_{n_j})$$

$$< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon \text{ for all } n_i, n_j \geq m. \text{ (by (1) and (2))}$$

Hence (x_{n_k}) is a Cauchy subsequence of (x_n).

$\therefore \overline{A}$ is totally bounded.

Problem 5. Let A be a totally bounded subset of **R**. Prove that $\overline{A}$ is compact.

Solution. Since A is totally bounded $\overline{A}$ is also totally bounded. VV.10

Also, since $\overline{A}$ is a closed subset of **R** and **R** is complete $\overline{A}$ is complete.

(refer theorem 3.4) page 8

Hence $\overline{A}$ is totally bounded and complete.

$\therefore \overline{A}$ is compact. (refer theorem 6.12)

Exercises

6.4 COMPACTNESS AND CONTINUITY

In this section we prove some results about continuous functions defined on a compact metric space. These are generalizations of the corresponding results for continuous real valued functions defined on any closed interval $[a, b]$.

Theorem 6.14. Let f be a continuous mapping from a compact metric space M_1 to any metric space M_2. Then $f(M_1)$ is compact.

(i.e.,) *Continuous image of a compact metric space is compact.*

Proof. Without loss of generality we assume that $f(M_1) = M_2$.

Let $\{G_\alpha\}$ be a family of open sets in M_2 such that $\cup G_\alpha = M_2$.

$\therefore \cup G_\alpha = f(M_1)$

$\therefore f^{-1}(\cup G_\alpha) = M_1$.

$\therefore \cup f^{-1}(G_\alpha) = M_1$.

Also since f is continuous $f^{-1}(G_\alpha)$ is open in M_1 for each α.

$\therefore \{f^{-1}(G_\alpha)\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover, say, $f^{-1}(G_{\alpha_1}), \ldots\ldots, f^{-1}(G_{\alpha_n})$.

$\therefore f^{-1}(G_{\alpha_1}) \cup f^{-1}(G_{\alpha_2}) \cup \ldots\ldots \cup f^{-1}(G_{\alpha_n}) = M_1$.

$$\therefore f^{-1}\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) = M_1.$$

$$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} = f(M_1) = M_2.$$

$\therefore G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}$ is a cover for M_2.

Thus the given open cover $\{G_\alpha\}$ for M_2 has a finite subcover.

$\therefore$ M_2 is compact.

com iff clos & bounded

Corollary 1. Let f be a continuous map from a compact metric space M_1 into any metric M_2. Then $f(M_1)$ is closed and bounded.

Proof. $f(M_1)$ is compact and hence is closed and bounded. [Th: 6.6] [Th: 6.4]

codomain R

Corollary 2. Any continuous real valued function f defined on a compact metric space is bounded and attains its bounds. bounding values

Proof. Let M be a compact metric space.

Let $f : M \to \mathbf{R}$ be a continuous real valued function.

Then $f(M)$ is a compact subset of $\mathbf{R}$.

$\therefore$ $f(M)$ is a closed and bounded subset of $\mathbf{R}$.

Since $f(M)$ is bounded f is a bounded function.

Now, let $a =$ $l.u.b.$ of $f(M)$ and $b =$ $g.l.b.$ of $f(M)$.

By definition of $l.u.b.$ and $g.l.b.$ $a, b \in \overline{f(M)}$.

But $f(M)$ is closed. Hence $f(M) = \overline{f(M)}$.

$\therefore$ $a, b \in f(M)$.

$\therefore$ There exist $x, y \in M$ such that $f(x) = a$ and $f(y) = b$.

Hence f attains its bounds.

Note. Corollary (2) is not true if M is not compact.

The function $f : (0, 1) \to \mathbf{R}$ defined by $f(x) = 1/x$ is continuous but not bounded

The function $g : (0, 1) \to \mathbf{R}$ defined by $g(x) = x$ is bounded having $l.u.b. = 1$ and $g.l.b. = 0$. However this function never attains these bounds at any point in $(0, 1)$.

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Theorem 6.15. Any continuous mapping f defined on a compact metric space (M_1, d_1) into any other metric space (M_2, d_2) is uniformly continuous on M_1.

Proof. Let $\varepsilon > 0$ be given. Let $x \in M_1$.

Since f is continuous at x there exists $\delta_x > 0$ such that

$$d_1(y, x) < \delta_x \Rightarrow d_2(f(y), f(x)) < \frac{1}{2}\varepsilon. \quad \text{..... (1)}$$

Now, the family of open balls $\{B(x, \frac{1}{2}\delta_x) / x \in M_1\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover say $B(x_1, \frac{1}{2}\delta_{x_1}), \ldots\ldots B(x_n, \frac{1}{2}\delta_{x_n})$.

Let $\delta = \min\{\frac{1}{2}\delta_{x_1}, \ldots\ldots \frac{1}{2}, \delta_{x_n}\}$

We claim that $d_1(p, q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon$.

Let $p \in B(x_i, \frac{1}{2}\delta_{x_i})$ for some i where $1 \leq i \leq n$.

$\therefore d_1(p, x_i) < \frac{1}{2}\delta_{x_i}$. f is continuous

$$\therefore d_2(f(p), f(x_i)) < \frac{1}{2}\varepsilon \quad \text{(by (1))} \quad \text{..... (2)}$$

Now, $d_1(q, x_i) \leq d_1(q, p) + d_1(p, x_i)$

$\leq \delta + \frac{1}{2}\delta_{x_i}$

$\leq \frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{x_i} = \delta_{x_i}$.

Thus $d_1(q, x_i) < \delta_{x_i}$

f continuous

$$\therefore d_2(f(q), f(x_i)) < \frac{1}{2}\varepsilon \quad \text{(by (1))} \quad \text{..... (3)}$$

Now, $d_2(f(p), f(q)) \leq d_2(f(p), f(x_i)) + d_2(f(x_i), f(q))$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \quad \text{(by (2) and (3))}$$

Thus $d_1(p, q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon$. single

This proves that f is uniformly continuous on M_1.

Note. The above theorem is not true if M_1 is not compact.

We have seen that if f is a continuous bijection then f^{-1} need not be continuous. Now we shall prove that if f is a continuous bijection defined on a compact metric space, then f^{-1} is also continuous.

Theorem 6.16 Let f be a 1 - 1 continuous function from a compact metric space M_1 onto any metric space M_2. Then f^{-1} is continuous on M_2. Hence f is a homeomorphism from M_1 onto M_2. $f: M_1 \to M_2$

Proof. We shall show that f^{-1} is continuous by proving that

F is a closed set in $M_1 \Rightarrow (f^{-1})^{-1}(F) = f(F)$ is a closed set in M_2. Thrm 4.3

Let F be a closed set in M_1.

Since M_1 is compact F is compact. (by theorem 6.4).

Since f is continuous $f(F)$ is a compact subset of M_2.

$\therefore$ $f(F)$ is a closed subset of M_2.

$\therefore$ f^{-1} is continuous on M_2.

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Solved problems 5 marks

Problem 1. Prove that the range of a continuous real valued function f on a compact connected metric space M must be either a single point or a closed and bounded interval.

Solution. Let $f : M \to \mathbf{R}$ be a continuou function.

Case (i) Suppose f is a constant function.

Then the range of f is a single point.

Case (ii) Suppose f is not a constant function.

Then the range of contains more than one point.

Since M is connected $f(M)$ is a connected subset of $\mathbf{R}$.

(refer theorem 5.5)

$\therefore$ $f(M)$ is an intervalin $\mathbf{R}$. (by theorem 5.3)

Also, since M is compact and f is continuous $f(M)$ is a compact subset of $\mathbf{R}$. (by theorem 6.14)

$\therefore$ $f(M)$ is a closed and bounded subset of $\mathbf{R}$.

Thus $f(M)$ is a closed and bounded interval of $\mathbf{R}$.

Problem 2. Prove that any continuous function $f : [a, b] \to \mathbf{R}$ is not onto.

Solution. Suppose f is onto. Then$f([a, b]) = \mathbf{R}$.

Now, since $[a, b]$ is compact and f is continuous, $f[a, b] = \mathbf{R}$ is compact which is a contradiction.

$\therefore$ f is not onto.

Exercises.

1. Prove that any continuous function from a compact metric space to any other metric space is a closed map.

2. Prove that any continuous function defined on a closed interval