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Unit - V

Sequence of function :

This Chapter deals with sequence $\{f_n\}$ of function which are real or complex valued functions having a common domain on the real line R or in the complex plane C.

For each x in the domain the sequence $\{f_n(x)\}$ whose terms are the corresponding values.

Let S denote the set of x for which this sequence converges

ie), $S = \{x / \{f_n(x)\}$ converges$\}$

Limit of a function :

The function f defined by the equation

$$f(x) = \lim_{n \to \infty} f_n(x), \quad x \in S$$

is called the limit function of the sequence $\{f_n(x)\}$ and we say that f_n converges point wise to f on the set S.

Note :

The limit function of the sequence of continuous functions need not be continuous.

For example, $f_n(x) = \dfrac{x^{2n}}{(1+x^{2n})}$, $n = 1, 2, \ldots$

$\lim_{n \to \infty} f_n(x)$ exists for every x.

The limit function $f(x)$ is given by

$$f(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ 1/2 & \text{if } |x| = 1 \\ 1 & \text{if } |x| > 1 \end{cases}$$

for each f_n is continuous, but f is not continuous.

Pointwise converges:

Let (f_n) be a sequence of function which converges point wise on a set S through to a limit function f. Then for each point x in S and for each $\varepsilon > 0$ there exists N such that $n > N$

$$\Rightarrow |f_n(x) - f(x)| < \varepsilon$$

Uniformly convergence:

A sequence (f_n) of function is said to be converge uniformly to a function f on a set S if for given $\varepsilon > 0$, there exists N such that

$$\Rightarrow |f_n(x) - f(x)| < \varepsilon \ \forall x \text{ in } S$$

Symballically, this can be written as $f_n \to f$ uniformly on S.

Uniformly bounded :

A sequence (f_n) of function is said to be uniformly bounded on S, if there exists a constant $M > 0$ such that $|f_n(x)| \le M \ \forall x$ in S and $\forall n$.

The number M is called a uniform bounded for (f_n).

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Theorem : 9.2

Assume that $f_n \to f$ uniformly on S. If each f_n is continuous at the point c of S, Then the limit function f is also continuous of c.

Proof :

Given $f_n \to f$ uniformly on S.

Then for given $\varepsilon > 0$ there exist M such that $n > m$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon/3 \ \forall x$ in $S \to (1)$

Let C be any point in the set S.

Given that f_n is continuous at c.

If c is an isolated point with S, the automatically f is continuous at c.

Suppose c is an ~~auto~~ accumulation point on S.

Since f_n is continuous at c, then there exist a neighbourhood $B(c)$ such that $x \in B(y) \cap S$ and $\Rightarrow |f_n(x) - f_n(c)| < \varepsilon/3$

In particular,

$$|f_m(x) - f_m(c)| < \varepsilon/3 \rightarrow (2)$$

Consider,

$$|f(x) - f(c)| = |f(x) - f_m(x) + f_m(x) - f_m(c) + f_m(c) - f(c)|$$

$$|f(x) - f(c)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(c)| + |f_m(c) - f(c)| \rightarrow (3)$$

In each point x in $B(c) \cap S$.

By (1) and (2) each term in the RHS is less than $\varepsilon/3$

Hence $|f(x) - f(c)| < \varepsilon$

$\therefore$ f is continuous at c.

Theorem: 9.3

Cauchy condition for uniform converges.

Let (f_n) be a sequence of function defined on a set S. There exist a function f such that $f_n \rightarrow f$ uniformly on S iff the following condition is satisfied

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For every $\varepsilon > 0$, there exist N Such that $n > N$ implies

$|f_n(x) - f_m(x)| < \varepsilon$ for every x in S.

Proof:

Let (f_n) be a sequence of function defined on S.

Suppose $f_n \to f$ uniformly on S.

Then for given $\varepsilon > 0$, there exist N such that $n > N$ implies

$|f_n(x) - f(x)| < \varepsilon/2, \forall x$ in S

In particular, $m > N$ implies

$|f_m(x) - f(x)| < \varepsilon/2$ every x in S

Consider,

$$|f_n(x) - f_m(x)| = |f_n(x) \cancel{+} f(x) - f(x) - f_m(x)|$$

$$\leq |f_n(x) + f(x)| + |f_m(x) - f(x)|$$

$$< \varepsilon/2 + \varepsilon/2$$

$|f_n(x) - f_m(x)| < \varepsilon \ \forall x$ in S

Hence Cauchy condition is satisfied

conversely,

Suppose that for every $\varepsilon > 0$ there exist an N such that $n > m$ and $m > N$ implies

$|f_n(x) - f_m(x)| < \varepsilon/2$ for every x in S

By Cauchy condition on sequence.

$\{f_n(x)\}$ converges for each x in S and

Suppose $\{f_n(x)\}$ converges to $f(x)$ say.

$\therefore \lim_{n\to\infty} f_n(x) = f(x)$ for each x in S $\to$ (*)

we shall show that $f_n \to f$ uniformly on S.

Let $\varepsilon > 0$ be given.

Choose N so that $n > N$ implies

$|f_n(x) - f_{n+k}(x)| < \varepsilon/2$ for every $k = 1, 2, \ldots$

and for every x in S.

$\therefore \lim_{k\to\infty} |f_n(x) - f_{n+k}(x)| = |f_n(x) - f(x)|$ (using (*))

ie) $|f_n(x) - f_{n+k}(x)| \leq \varepsilon/2 \ \forall x$ in S

$\therefore f_n \to f$ uniformly on S.

Uniform convergence of infinite series of function.

Given a sequence $\{f_n\}$ of function defined on a set S. we can form a series $\sum f_n(x)$

For each x in S we define,

$$S_n(x) = \sum_{k=1}^{n} f_k(x)$$

If there exist a function f such that $S_n \to f$ uniformly on S,

we write $\sum f_n(x) = f(x)$ uniformly on S.

State & prove

Theorem : 9.5 (Criteria)

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Cauchy condition for uniform converges of series:

The infinite series $\sum f_n(x)$ converges uniformly on S iff for every $\varepsilon > 0$ there exist N such that $n > N$ implies.

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \varepsilon, \quad p = 1, 2, \ldots$$

Proof:

Suppose that $\sum f_n(x)$ converges uniformly on S.

Define $S_n(x) = \sum_{k=1}^{n} f_k(x)$

By hypothesis

$S_n \to f$ uniformly on S.

By cauchy condition on sequence of functions for every $\varepsilon > 0$ there exist N such that $n > N$ and $p = 1, 2, \ldots$ implies

$$|S_n(x) - S_{n+p}(x)| < \varepsilon$$

ie) $$\left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{n+p} f_k(x) \right| < \varepsilon$$

ie) $$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \varepsilon$$

Conversely,

Suppose for given $\varepsilon > 0$ there exist N such that $n > N$ and $p = 1, 2, \ldots$ implies

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \varepsilon \quad \forall\ n > N\ p = 1, 2, \ldots$$

$$\left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{n+p} f_k(x) \right| < \varepsilon \quad \forall\ n > N\ p = 1, 2, \ldots$$

$$\left| S_n(x) - S_{n+p}(x) \right| < \varepsilon \quad \forall\ n \geq N\ p = 1, 2, \ldots$$

By cauchy condition for sequence of function

$S_n \to f$ uniformly on S.

Thus the given series $\sum f_n(x)$ converges uniformly on S.

Theorem: 9.6

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Weirstrass M-Test

Let (M_n) be a sequence of non-negative numbers such that $0 \leq |f_n(x)| \leq M_n$, $n = 1, 2, \ldots$ and for every x in S. Then $\sum f_n(x)$ converges uniformly on S if $\sum M_n$ converges.

Proof:

Let (M_n) be a sequence of non-negative number such that $0 \leq |f_n(x)| \leq M_n$ $n = 1, 2, 3, \ldots$

now $|f_n(x)| \leq M_n$

$$\therefore \left| \sum_{k=n+1}^{n+p} f_k(x) \right| \leq \left| \sum_{k=n+1}^{n+p} M_k \right| \rightarrow (1)$$

Given (M_n) is a sequence of non-negative number and $\sum M_n$ converges.

$\therefore$ By cauchy criterion

for given $\varepsilon > 0$ there exist N such that

$$\left| \sum_{k=n+1}^{n+p} M_k \right| < \varepsilon \quad \forall \; n > N, \; p = 1, 2, \dots$$

From (1) $\Rightarrow \left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \varepsilon \quad \forall \; n > N, \; p = 1, 2, \dots$

By cauchy condition for uniform convergence of Series, $\sum f_n(x)$ converges uniformly on S.

Theorem : 9.7

5m Assume that $\sum f_n(x) = f(x)$ uniformly on S, if each f_n is continuous at a point x_0 of S, then f is also continuous at x_0.

Proof :

given $\sum f_n(x) = f(x)$ uniformly on $S_n \to f$ uniformly on S.

Also given each f_n is continuous at x_0.

$$\lim_{x \to x_0} f_n(x) = f(x_0)$$

we have to prove that

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Since $S_n \to S_f$ we have,

$$\lim_{n\to\infty} S_n(x) = f(x)$$

consider, $\lim_{x\to x_0} f(x) = \lim_{x\to x_0}\left(\lim_{n\to\infty} S_n(x)\right)$

$$= \lim_{n\to\infty}\left(\lim_{x\to x_0} S_n(x)\right)$$

$$= \lim_{n\to\infty}\left(S_n(x_0)\right) \quad (\because \text{each } f_n \text{ is continuous at } x_0)$$

$\lim_{x\to x_0} f(x) = f(x_0)$ $\left(\because \lim_{n\to\infty} S_n(x) = f(x)\right)$

Hence f is continuous at x_0.

Boundedly convergence :

A sequence of function $\{f_n\}$ is said to be boundedly convergent on T if $\{f_n\}$ is a pointwise convergence and uniformly bounded.

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Theorem : 9.11 Arzela thm

model

Let $\{f_n\}$ be boundedly convergence sequence on $[a, b]$. Assume that each $f_n \in R$ on $[a, b]$ and the limit function $f \in R$ on $[a, b]$. Assume also that there exist a partition P of $[a, b]$ Say $P = \{x_0, x_1, \ldots x_n\}$ such that on every Subinterval $[c, d]$ not containing say of the point x_k, the sequence $\{f_n\}$ converges uniformly to f. Then we have

$$\lim_{n\to\infty} \int_a^b f_n(t)\, dt = \int_a^b \lim_{n\to\infty} f_n(t)\, dt = \int_a^b f(t)\, dt$$

Proof :

Given, $\{f_n\}$ be a boundedly convergent Sequence on $[a, b]$.

Since the limit function $f \in R$ on $[a, b]$, it is bounded.

Also, since f_n is boundedly convergens and f is a bounded function.

there exist a positive integer M such that $|f(x)| \leq M$ and $|f_n(x)| \leq M$ $\forall$ x in $[a,b]$ $\forall$ $n \geq 1$

Let $\varepsilon > 0$ be such that $\|P\| > 2\varepsilon$ and Let $h = \varepsilon/2m$.

where m is the number of subintervals of P.

Consider a new partition P' of $[a, b]$ given by:

$$P' = \{x_0, x_0+h, x_i-h_i, x_i+h, \ldots x_{m-1}+h, x_{m-1}-h, x_m\}$$

Since both $f, f_n \in R$ on $[a,b]$ they are integrable function on $[a,b]$

$\therefore f - f_n$ is integrable on $[a,b]$

Thus $|f - f_n|$ is integrable on $[a,b]$ and

$|f - f_n| \leq |f| + |f_n| < M + M$

$\therefore |f - f_n| < 2M$

Hence $|f - f_n|$ is bounded by $2M$

Consider the partition,

$[x_0, x_0+h], [x_1-h, x_1+h], \dots [x_{m-1}+h, x_m]$ of

$[a,b]$ the sum of integrals of $|f - f_n|$ taken over the intervals is at most $2M$.

ie) $\sum \int |(f - f_n)(x)|\, dx = 2M\varepsilon, \; x \in [x_0, x_m]$

The remaining ~~partition~~ portion of $[a,b]$ is the union of finite numbers of closed intervals named as S and in each of which f_n converges uniformly to f.

$\therefore$ There exist an integers N such that

$n > N$ implies $|f(x) - f_n(x)| < \varepsilon$

Hence the sum of the integrals of $f - f_n$ over the intervals S is atmost $\varepsilon(b-a)$

So; $\int_a^b (f(x) - f_n(x))\, dx \leq (2M + b - a)\varepsilon$

whenever $n \geq N$

Since $\varepsilon > 0$ is arbitrary

$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$

Theorem : 9.13

Uniform convergence of Derivatives.

Assume that each term if $\{f_n\}$ is a real-valued function having a finite derivative at each point of an open interval (a, b). Assume that for at least one point x_0 in (a,b) the sequence $\{f_n(x_0)\}$ converges. Assume further that there exist a function g such that $f_n' \to g$ uniformly on (a,b). Then

a) There exist a function f such that $f_n \to f$ uniformly on (a, b)

b) For each x in (a,b) the derivative $f'(x)$ exists and equals $g(x)$.

Proof:

Let each f_n be differentiable on (a,b)

Assume $f_n' \to g$ uniformly on (a,b)

we prove that, $f_n \to f$ uniformly and

f' exist and $f'(x) = g(x)$

fix, $c \in (a,b)$ and

Define, $g_n(x) = \left\{ \dfrac{f_n(x) - f_n(c)}{x-c} \right., x \neq c$

This sequence $g_n(x)$ depends on c

Since $f_n' \to g$ uniformly and

$f_n'(c) \to g(c)$

the sequence $g_n(x)$ converges

we now prove that $g_n(x)$ converges uniformly.

uniform convergence

if $x \neq c$,

$$g_n(x) = \frac{f_n(x) - f_n(c)}{x-c}, \quad g_m(x) = \frac{f_m(x) - f_m(c)}{x-c}$$

$$g_n(x) - g_m(x) = \frac{f_n(x) - f_n(c)}{x-c} - \frac{f_m(x) - f_m(c)}{x-c}$$

$$= \frac{|f_n(x) - f_m(x)| - |f_n(c) - f_m(c)|}{x-c}$$

Define $h(x) = f_n(x) - f_m(x)$

Then $g_n(x) - g_m(x) = \dfrac{h(x) - h(c)}{x-c}$

By mean value theorem

$$\frac{h(x) - h(c)}{x-c} = h'(x_1) \text{ for some } t \in (a,b)$$

So, $g_n(x) - g_m(x) = f_n'(x_1) - f_m'(x_1)$

$\therefore f_n' \to g$ uniformly, g_n is uniformly Cauchy

So g_n converges uniformly

Part (a) : Uniform convergence of f_n

Let $x_0 \in (a,b)$

$c = x_0$

Define

$$g_n(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0}$$

$$\Rightarrow f_n(x) = (x - x_0)\, g_n(x) + f_n(x_0)$$

Then $f_m(x) = (x - x_0)\, g_m(x) + f_m(x_0)$

$$f_n(x) - f_m(x) = (x - x_0)\left[g_n(x) - g_m(x)\right] + \left[f_n(x_0) - f_m(x_0)\right]$$

$\because$ g_n is uniformly convergent and $f_n(x_0)$ is convergent

by Cauchy condition, $f_n \to f$ uniformly

Part (b) :

Show $g_n \to g$ is continuous

Again define,

$$g_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c}, & x \neq c \\ f_n'(c), & x = c \end{cases}$$

Then

$$\lim_{x \to c} g_n(x) = \lim_{x \to c} \frac{f_n(x) - f_n(c)}{x - c}$$

$$= f_n'(c)$$

$$= g_n(c)$$

So each g_n is continuous at c.

$\because$ $g_n \to g$ uniformly the limit function g is also continuous

Let $\varphi(x) = \lim_{n\to\infty} g_n(x)$

Then φ is continuous

when $x \neq c$,

$$\varphi(x) = \lim_{n\to\infty} \frac{f_n(x) - f_n(c)}{x-c}$$

$$= \frac{f(x) - f(c)}{x \neq c} \quad \left(\text{mean value Thm}\right)$$

$$\Rightarrow \lim_{x\to c} \varphi(x) = f'(c)$$

But $\varphi(c) = \lim_{n\to\infty} g_n(c) = \lim f_n'(c) = g(c)$

So $f'(c) = g(c)$

Since c is arbitrary $f'(c) = g(x)\ \forall\ x \in (a,b)$.

Note :

The above theorem can be reformulated in the series as follows

Theorem : 9.14

Lemma : Theorem 9.13

Assume that each f_n is a real valued function defined on (a,b) such that the derivative $f_n'(x)$ exists for each x in (a,b). Assume that for atleast one point x_0 in (a,b) the series $\sum f_n(x_0)$ converges. Assume that there exist a function g such that $\sum f_n'(x) = g(x)$. Then

a) $\exists$ a function f such that

$\sum f_n(x) = f(x)$

b) If $x \in (a,b)$ the derivative $f'(x)$ $\exists$ a equals $\sum f_n'(x)$

(X) **Theorem : 9.15** ✓ State & Prove

8m

Dirichlet's test for uniform convergence

Let $f_n(x)$ denote the n^{th} partial sum of the series $\sum f_n(x)$ where each f_n is a complex valued function defined on a set S. Assume that $\{f_n\}$ is uniformly bounded on S. Let $\{g_n\}$ be a sequence of real valued function such that $g_{n+1}(x) \leq g_n(x)$ for each x in S and for every $n = 1, 2, \cdots$ and assume that $g_n \to 0$ uniformly on S. Then the series $\sum f_n(x) \cdot g_n(x)$ converges uniformly on S.

Proof :

Given (f_n) is uniformly bounded

(g_n) is an increasing sequence of real valued function.

$(g_n) \to 0$ uniformly on S.

To prove : $\sum f_n(x)\, g_n(x)$ converges uniformly on S.

Let $(S_n(x))$ be a sequence of partial Sum for $\sum f_n(x)\, g_n(x)$

Then $S_n(x) = \sum_{k=1}^{n} f_k(x)\, g_k(x)$

$$S_n(x) = \sum_{k=1}^{n} F_k(x)\left[g_k(x) - g_{k+1}(x)\right] + g_{n+1}(x)\, F_n(x)$$

If $n > m$ we can write

$$\left[S_n(x) - S_m(x)\right] = \left[\sum_{k=1}^{n} f_k(x)\, g_k(x)\right] - \left[\sum_{k=1}^{n} f_k(x)\, g_k(x)\right]$$

$$= \sum_{k=1}^{n} F_k(x)\left[g_k(x) - g_{k+1}(x)\right] + g_{n+1}(x)\, F_n(x) - g_{m+1}(x)\, F_m(x)$$

Since (f_n) is uniformly bounded $\exists\ m > 0$ Such that $|f_n(x)| \leq m$ we have

$$|S_n(x) - S_m(x)| \leq M \sum_{k=m+1}^{n} |g_k(x) - g_{k+1}(x)| + M|g_{n+1}(x)| + M|g_{m+1}(x)|$$

$$|S_n(x) - S_m(x)| < 2M\, g_{m+1}(x) \longrightarrow 1$$

Given $g_n \to 0$ uniformly on S

For given $\varepsilon > 0\ \exists\ N$ such that $|g_n(x)| < \varepsilon$ $\forall\ n > N$.

From 1 $|S_n(x) - S_m(x)| < 2M\varepsilon$

Since $\varepsilon > 0$ is arbitrary by cauchy condition the sequence of partial sums (S_n) converges uniformly.

The given series $\sum f_n(x)$ converges uniformly on S.

Theorem : 9.12

Arzela

Assume that (f_n) is boundedly convergent on $[a, b]$ and suppose each f_n is Riemann integral on $[a, b]$. Assume also that the limit function f is Riemann integrable on $[a, b]$. Then

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n \to \infty} f_n(x)\, dx = \int_a^b f(x)\, dx$$

Proof

Lemma : Thm 9.11