UNIT V

GRAPH THEORY

5.1 GRAPHS AND BASIC TERMINOLOGIES

A graph is a mathematical concept which can be used to model many concepts from –the real world.

A **graph** consists of a pair of sets, represented as *G* = (V, *E*), where V is a non-empty set of **vertices** (also called **nodes**) and E is a set of **edges** (sometimes called **arcs**).

An edge can be represented as a pair of nodes (u,v) indicating an edge from node u to node v.

Two vertices/nodes *x* and y of *G* are *connected* if there is an edge *xy* between them, and these vertices are then called *adjacent* or *neighbour* vertices/nodes. Here, the nodes *x* and y are called the *endpoints* of the edge.

x  y

In a graph *G*, a node which is not adjacent to any other node is called an *isolated node*.

A graph is *finite* if it has a finite number of vertices and a finite number of edges, otherwise it is infinite.

If G is finite, *G(*V)*denotes* the number of vertices in *G* and it is called the *order* of *G.*

Similarly, *E(G)* denotes the number of edges in *G* and it is called the *size* of G.

The graph shown in figure 5.1has four vertices a,b, c and d.

(a,b) is a pair of vertices which are connected, and this connectivity represents an edge between them. Now a and*b* are the *end points* of the edge (*a,* b).

*N*eighbours of vertex *a* in this graph are *b* and *c* as there are edges from *a* to *b* and *a* to *c.*

Vertex *d* is the *isolated* vertex, as it is not adjacent to any other vertices.

It is an example of finite graph and its Order of G is 4 as V = {a, b, c, d}.

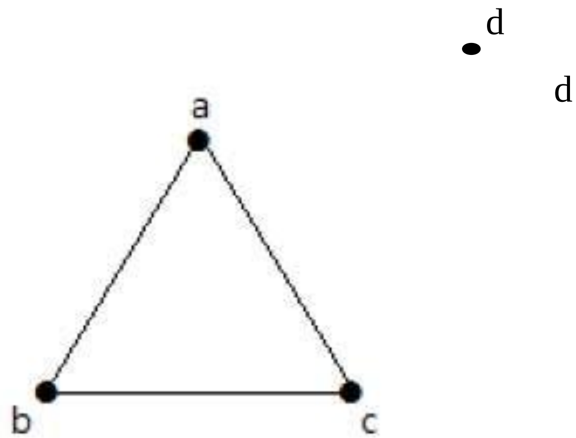


Figure 5.1 A graph with isolated vertex

1. Undirected and Directed Graphs

Graphs may be *directed* or *undirected*.

A graph is directed (or *digraph*) when direction of edge from one vertex to another is defined, otherwise it is an undirected graph.

Undirected edge between vertices *u* and v is expressed as *(u,* v).

Directed edge between vertices *u* and *v* is expressed as *<u,v>*

A simple directed graph is shown in Figure 5.2

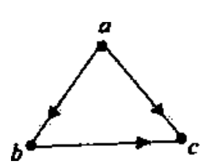


Figure 5.2 Simple Directed Graph

2. Weighted Graph

A weighted graph associates a value (*weight*) with every edge in the graph.

In other words, when a weight (may be cost, distance, etc) is associated with each edge of a graph, then it is called as weighted graph, otherwise **unweighted** graph.

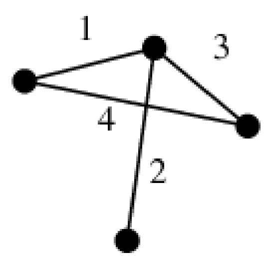


Figure 5.3 *edge-labeled graph*

3. Self-Edge or self –Loop

In graph theory, a **loop** (also called a **self-loop** or a "buckle") is an edge that connects a vertex to itself. A simple graph contains no loops.

A graph with self loop is shown Figure 5.4.

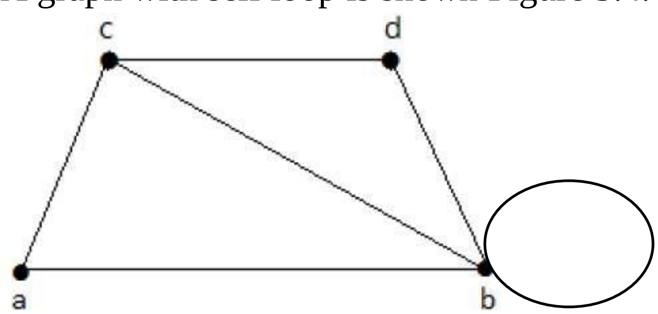
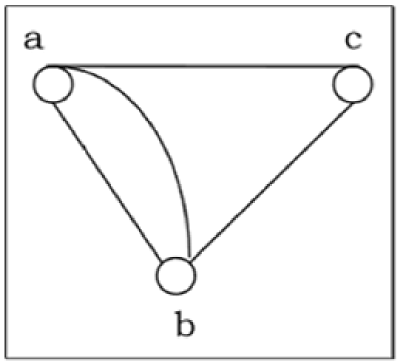


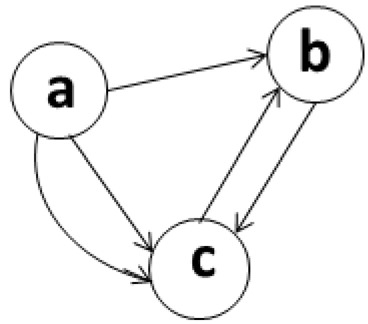
Figure 5.4

4. Multiple or parallel Edges

If a pair of nodes is joined by more than one edge, then such edges are called *multiple* or *parallel* edges.

In *undirected* graph, two edges(v_i, v_j) and $(v_p = v_q)$ are parallel edges if$v_i = v_p and\ v_j = v_q$.





(a) Undirected graph with parallel edges

(b) Directed graph with parallel edges

Figure 5.5

In the directed graph, the edges between vertices a and b parallel edges. The edge pair between the vertices b and c are not *parallel edges*, since the directions of the edge pair are *opposite.*

5. Path in a Graph

A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. Clearly, vertices as well as edges may be repeated in a path.

A path from *u* to w is a sequence of edges $(u, v_1), (v_1, v_2)$ ...., (v_{k-1}, w)connecting *u* with w.

A path may be termed as *walk* also.

The number of edges in a path is termed as its *length.*

For example,*(a, b*), *(b,b*), (*b,* d), *(d, c)*, *(c, b*), (*b,* d) is one path of the graph as shown in Figure 5.4. and its length is 6.

A path with no repeated vertex is called a *simple path.*

In Figure 5.4. (a, *b)*, (*b, d), (d, c) is simple path.*

A path with no repeated edge is termed as *trail*.

In a closed trail, the first and the last vertices are same.

A closed path is a path that starts and ends at the same point, otherwise the path is open. Edge repetition is allowed on the closed path.

In Figure 5.4. (a, *b), (b, d), (d, c),(c,a) is a closed path.*

A **Cycle** (**circuit**/**tour**) is a closed path of non-zero length that does not contain any repeated edges. Vertices other than the *end* (i.e., *start)* vertex may also be repeated.

In Figure 5.4, *(a, b),* (*b, b), (b, d), (d, c),(c, a)* is a cycle.

A *simple cycle* is a cycle that does not have any repeated vertex except the first and the last vertex. *(a, b),(b, d), (d, c),(c, a)* is an example of simple cycle.

A graph without cycles is called *acyclic.*

A *tree* is an acyclic and connected graph.

A *forest* is a set of trees.

6. Connected Graph

A graph is called *connected* if and only if for any pair of nodes *u,* v, there is at least one path between *u* and v. Otherwise, it is disconnectecd.

Clearly, the graph in Figure 5.4 is a connected undirected graph, whereas the graph given in Figure 5.1 is disconnected.

7. Types of Connectivity in Graphs

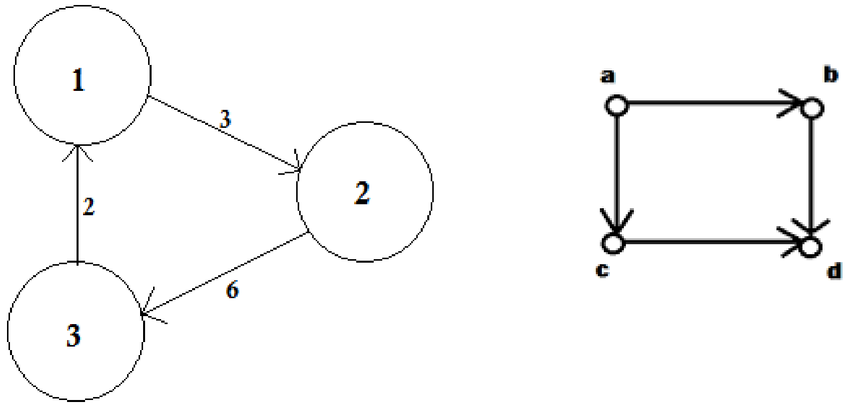
A connected graph must have at least two vertices.

A graph is *strongly connected* if and only if every pair of vertices in the graph are reachable from each other. i.e., if there are paths in both directions between any two vertices.

Otherwise, the graph is of *weakly* or *unilaterally* connected.

The graph in Figure 5.6(a) is an unilaterally connected graph, as it has a path from *a* to *c* but no path exists from *c* to *a,* and so on.

A graph is *strictly weakly* connected if it is not unilaterally connected. Thus, a strictly weakly connected graph may have many *sources* and *sinks* (destinations). The graph given in Figure 5.6(b) is an example of *strictly* weakly connected graph.



(a) (b)

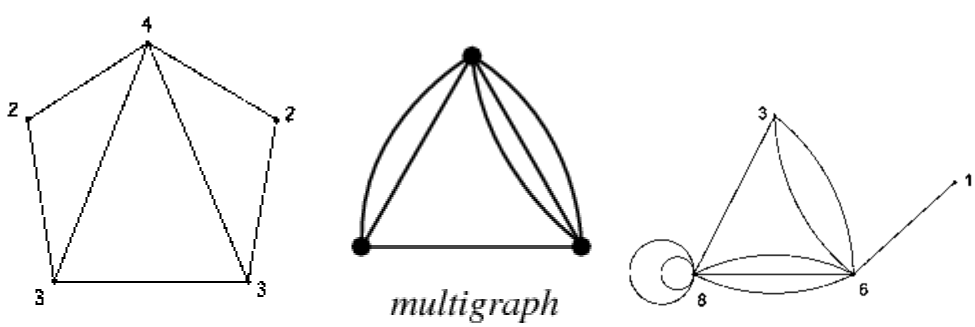
Figure 5.6

8. Simple Graph, Multi – Graph, and Pseudo-Graph

A directed or undirected graph which has neither self-loops nor parallel edges is called *simple graph.*
However, cycle(s) is (are) allowed in a simple graph.
Further, a simple graph may contain *isolated* vertex also.

The graph as shown in Figure 5.7(a) is a simple connected graph, since it has no self loop and parallel edges.

Further, the graph in Figure 5.6 is a *simple directed* graph, as it has no self-loop or parallel edges. On the other graph in Figure 5.1 is a *simple disconnected* graph.



(a) (b) (c)

Figure 5.7

Any graph (directed or undirected) which contains some parallel edges is called a multigraph. In multi-graph, no self-loop is allowed but cycle may be present.

Figure 5.7 (b) is an example of multi-graph, since it has parallel edges but *no* self loop.

A directed or undirected graph in which self-loop(s) and parallel edge(s) are allowed is called a *pseudo-graph.* Figure 5.7 (c) is an example of *pseudo-graph.*

9.Degree of vertex

The degree of a vertex of an undirected graph is the number of edges incident on itcounting self loop twice. The degree of a vertex G is denoted by $deg_G(v)$.

For example In the undirected graph in Figure 5.4, the degree of *a* [i.e., $deg_G(a)$ is 2, degree of b [i.e., $deg_G(b)$] is 5 (since there is a self-loop at *b),* degree of *c* is 3 and degree of d is 2.

In directed graph *G,* we consider two types of degrees of vertices: (a) ***in-degree*** *and (b)* ***out-degree****.*

The in-degree of a vertex v of G, denoted by $deg_G-(v)$, is number of edges moving into that vertex.

The *out-degree* of v, denoted by $deg_G^+(v)$is the the number of edges moving out from that vertex.

The sum of the in-degree and the *out-degree* of a vertex is called the *total degree* of that vertex.

For example:

In the directed graph in figure 5.6(a),the in-degree of 1,$deg_G-(1)$ is 0(zero) and but its out-degree, $deg_G^+(v)$ is 1.

Hence the total degree of 1 is $deg_G-(1) + deg_G^+(1) = 2$.

A vertex with zero in-degree is called a *source vertex* and a vertex with zero out-degree is called *sink* vertex.

A vertex of degree 0(zero) is called isolated vertex.

A vertex is pendant vertex if and only if its degree is 1.

The vertex d in Figure 5.1 is an isolated vertex, as its degree is zero, whereas the vertex 1 of the graph in Figure 5.7(c) is a pendant vertex (because its degree 1).

10. Degree Sequence of a Graph

Let G be a graph with vertices v_1, v_2,v_3, …v_n. The monotonically increasing sequence (d_1, d_2, d_3, …d_n), where $d_i = deg_G(v_i)$ is called the degree sequence of the graph G.

The degree sequence of the graph in figure 5.4 is (2,2,3,5).

Note :

The degree of a graph G is the *maximum* of the degrees of all nodes in G.

If the number of edges $m = O(n)$ (where *n* is the number nodes in the graph), then the graph is said to be *sparse*.

If *m* is larger than linear order of *n*, i.e., $m = 0(n^2)$ (but as long as there are no multiple edges), then the graph is called *dense*.

Theorem 5.1

A simple graph with n ≥ 2vertices contains atleast two vertices of the same degree.

Proof :

Let *G* be a simple graph with n ≥ 2vertices.

Since G is a simple graph ,it has no loop and parallel edges.

We know that, the degree of a vertex of a simple graph G on n vertices cannot exceed n-1.

So, degree of each vertex is ≤ 1.

Assume that all the vertices of *G* have distinct degrees.

Thus, the degrees, 0, 1, 2, 3,….,n-1,are possible for n vertices of G.

Let u be the vertex with degree 0.Clearly,u is the isolated vertex.

Let v be the vertex with degree n-1,then v must have n-1 adjacent vertices.

In fact, it is possible if the vertex v is adjacent to each vertex of the graph G, it is also adjacent to u. But it is assumed that u is an isolated vertex. i.e., it is not adjacent with any vertex of G.

Hence ,either u is not an isolated vertex or the degree of v is not n-1.

So, contradiction occurs on the assumption of different distinct degrees of vertices of G.

Thus, the contradiction proves that a simple graph contains at least two vertices of same degree.

Note :

The above theorem can be clearly understood by taking some examples.

(i) Suppose, the simple graph G has two vertices v_1 and v_2.

Since it is a simple graph, G has no loop or parallel edges.

First, consider that both the vertices are isolated.

Hence, $deg_G(v_1) = deg_G(v_2) = 0$.

So, both the vertices have same degree 0.

Second, suppose that they are adjacent to each other(but no parallel edges).

Hence, $deg_G(v_1) = deg_G(v_2) = 1$.

So, both the vertices have same degree 1.

(ii) Suppose, the simple graph G has three vertices v_1 , v_2 and v_3.

The graph G is a simple graph, so it has no loop or parallel edges.

First, consider that all the vertices are isolated.

Hence, $deg_G(v_1) = deg_G(v_2) = deg_G(v_3) = 0$.

So, at least two vertices (here all the vertices)have same degree 0.

Second, suppose, one vertex is isolated and the remaining two are adjacent to each other.

Then, degree sequence is(0,1,1) and it means at least two vertices out of three have same degree.

Third, suppose G has no isolated vertices out of three.

Then, degree sequence is (1,1,2) and it means at least two vertices out of three have same degree.

Hence, it is true for a simple graph with any number of vertices.

The above theorem is true for both directed as well as undirected graphs.

Theorem 5.2(The Handshaking theorem)

If G=(V,E) is a graph with e number of edges, then

$$\sum_{v\in V} deg_G(v) = 2e$$

i.e., the sum of degrees of the vertices of G is always even.

For directed graph,

$$\sum_{v\in V} deg_G(v) = \sum_{v\in V} deg^{-}(v) + \sum_{v\in V} deg^{+}(v)$$

i.e., the sum of degrees of the vertices is the sum of the in-degrees and the out-degrees of the vertices.

Proof :

Let G be an undirected graph.

The degree of a vertex of G is the number of edges incident with that vertex.

Now, every edge is incident with exactly two vertices.

Hence, each edge gets counted twice, one at each end.

Thus, the sum of the degrees equals twice the number of edges.

Let G be a directed graph.

Then in-degree and out-degree of each vertex of G are considered.

However, the sum of the in-degree and out-degree of a vertex is the total degree of that vertex.

Further, every edge is incident with exactly two vertices. So, here also, each edge gets counted twice: one as in-degree and the other as out-degree.

Thus, the sum of the degrees(in-degrees and out-degrees) of all the vertices equals twice the number of edges.

Note:

(i) The name of this theorem is handshaking because if several people shake hands, the total number of hands involved must be even(since for every handshaking, two hands are required).

(ii) This theorem applies even if multiple edges and self-loops are present in graph.

(iii) The theorem it is true for both connected and disconnected graphs.

(iv) If sum of degrees of the vertices of a graph is given, then the number of edges present in that graph can be computed. But the reverse is not possible.

Corollary :

In a graph, total number of odd-degree vertices is even.

Proof :

Let G=(V,E) be a graph, where K_1 and K_2 are the set of vertices with odd degree and even degree, respectively.

Now, $\sum_{v_i \in V} degG^{(v_i)} = \sum_{v_i \in K_1} degG(v_i) + \sum_{v_i \in K_2} degG(v_i)$

$$2e = \sum_{v_i \in K_1} degG(v_i) + \sum_{v_i \in K_2} degG(v_i)$$

[Since sum of the degree of vertices is twice the number of edges(e), and it is always even.]

Further, sum of the even-degree vertices is even.

i.e., $\sum_{v_i \in K_2} degG(v_i)$ is even.

Clearly,

$$\sum_{v_i \in K_1} degG(v_i) \text{is even.}$$

i.e., the sum of the odd-degree vertices is also even, Again,$\sum_{v_i \in K_1} degG(v_i)$ is even only if number of vertices of K_1 is even.

Hence, the number of odd degree vertices is even.

[For example, suppose three vertices contain odd degrees 1,3,5respectively.Clearly,there sum may not be even, since number of vertices is 3 which is an odd number.]

Note:

The sum of two numbers(say,n_1 and n_2)gives even if both of n_1 and n_2 are either odd or even. i.e., odd+odd=even, even+even=even.

Theorem 5.3

If G=(V,E) be a directed graph with e number of edges, then

$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v)$

i.e., the sum of the out-degrees of the vertices of G equals the sum of the in-degree of the vertices, which equals the number of edges in G.

Proof :

Any directed edge of G contributes 1 out-degree and 1 in-degree. Also, a self-loop contributes two degrees(1 out-degree and 1 in-degree).

Hence, the theorem is proved.

Example:

In the directed graph in Figure 5.6 (a),the in-degree of1, $degG^-(1)$ is 1 and its out-degree $degG^+(1)$ is 1.

Hence, the total degree of 1 is 2.

$$degG(1) = de\,gG^-(1) + de\,gG^+(1) = 1 + 1 = 2,$$
$$degG(2) = degG - (2) + de\,gG^+(2) = 1 + 1 = 2,$$
$$degG(3) = de\,gG^-(3) + de\,gG^+(3) = 1 + 1 = 2.$$

Hence, $deg G^{-}(1) + deg G^{-}(2) + deg G^{-}(3) = 1 + 1 + 1 = 3 =$ e(number of edges) and

$deg G^{+}(1) + deg G^{+}(2) + deg G^{+}(3)$=1+1+1=3=e(number of edges).

Example 5.1:

Show that the degree of a vertex of a simple graph G on n vertices cannot exceed n-1.

Solution :

Let v be a vertex of G.

Since G is simple, no multiple edges or self-loops are allowed in G.

Thus v can be adjacent to at most all the remaining n-1 vertices of G.

Hence, v may have maximum degree n-1 in G.

If the degree of v becomes more than(n-1),then there must have self-loops or parallel edges in the graph, which is not allowed in simple graph.

So, the degree of a vertex $v \in V(G)$ in a simple graph lies in the range $0 \leq degG(v) \leq n - 1$.

In particular, it is 0 if the vertex is isolated.

Note :

The above inequality is true for both directed and undirected simple graphs.

Example 5.2 :

Show that the maximum number of edges in a simple undirected graph with n vertices is n(n-1)/2.

Solution :

By the handshaking theorem, we know

$$\sum_{v \in V} deg_G(v) = 2e$$

where e is the number of edges with n vertices in the graph G.

This implies

$d(v_1) + d(v_2) + d(v_3) + \cdots + d(v_n) = 2e$ ……. (5.1)

Since maximum degree of each vertex in a simple graph can be (n-1).

Therefore, Eq. (5.1) can be written as

(n-1)+(n-1)+…+ up to n terms(considering maximum degree for each vertex)

=n(n-1)

=2e

Hence,

e(maximum number of edges in a simple graph with n vertices)=n(n-1)/2.

Note :

Maximum number of edges in a simple directed graph G is 2n(n-1)/2 =n(n-1),since in a simple directed graph, edges with opposite direction between any pair of vertices are allowed.

Example 5.3 :

For a simple graph with n vertices, what is the minimum number of edges required to ensure that the graph is connected?

Solution :

Let S c V be a set of vertices for which each vertex in S has degree 0.

If S has just one vertex(the minimum case of a disconnected graph),then(n-1) edges are possible between S and (V-S).

Therefore, the maximum possible number of edges in a disconnected graph is

$n(n-1)/2-(n-1) \quad =(n-1)[(n/2)-1]$

$= (n-1)(n-2)/2$

Clearly, the minimum number of edges in a connected graph is

$= 1+(n-1)(n-2)/2$

$=[2 +(n^2-3n+2)]/2.$

$= (n^2-3n+4)/2$

Note : To check the **existence of a graph** when its **degree sequence** is given.

1. If the sum of the degrees of the vertices of the graph is not even, then graph corresponding to the given degree sequence cannot be drawn (application of handshaking theorem).

2. If the total number of odd degree vertices (counted from the given degree sequence) is odd, then graph corresponding to the given degree sequence cannot be drawn.

Hence, for the existence of any graph G, the number of odd-degree vertices must be even, and this point can be applied only for confirmatory checking. i.e., it is not compulsory to consider.

Now, if both the above-mentioned conditions are false(i.e., when the sum of the degrees is even and the number of odd-degree vertices is also even, then it is certainly possible to draw one graph, but it may not be possible to draw a simple graph following the given degree sequence.

For checking the existence of a simple graph, we must concentrate on its properties. Some examples on degree sequence are given below.

Example 5.4 :

Is there a simple graph corresponding to the following degree sequences?

(a) (1,1,2,3)

(b) (2,2,4,4)

Solution :

(a) The total number of odd-degree vertices in a graph is even.

The number of odd-degree vertices is 3.

Hence, no graph corresponding to this degree sequence can be drawn.

Sum of degrees=1+1+2+3=7, which is odd.

By handshaking theorem, the sum of degrees of any simple graph must be even.

Hence, no graph exists for this case.

(b) The sum of the degree of the vertices is 12 which is even.

Also, the number of the odd-degree vertices is 0 which is even.

So, a graph can be drawn, using the given degree sequence.

Now, let us check if any simple graph is possible to draw or not.

The number of vertices is 4.

However, the degree of any vertex in a simple graph G on n vertices cannot exceed n-1, the degree of any vertex cannot be 4.

Hence, no simple graph corresponding to the given degree sequence can be drawn.

Example 5.5 :

Does there exist a simple graph with seven vertices having degrees(1,3,3,4,5,6,6)?

Solution:

The sum of the degrees of the vertices is 1+3+3+4+5+6+6=28 and it is an even number.

Also, the number of odd-degree vertices is even.

So, the graph corresponding to the given degree sequence exists.

Now, let us check whether any simple graph exists or not.

Assume that it exists. Here, two vertices out of seven have degree 6.So,each of these two vertices is adjacent to the rest six vertices of the graph. Accordingly, the degree of each vertex should be at least 2. i.e., it may not be 1.But in the degree sequence, no vertex with degree 2 is provided. Moreover, a vertex with degree 1 is given.

Therefore, we arrive at a contradiction in out assumption. Thus no simple graph, following the given degree sequence, can be drawn.

Example 5.6 :

For the graph G as shown in figure 5.8, write the degree sequence of G.

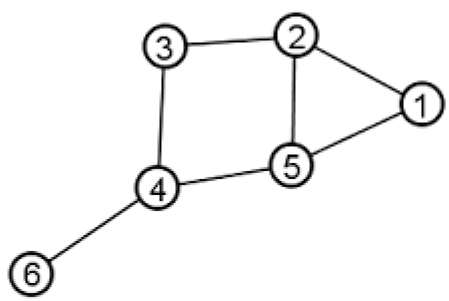


Figure 5.8

Hence, find the number of odd-degree vertices and the number of edges in the graph G.

Solution:

The degree sequence is given as {3,3,3,2,2,1}.

Hence, the number of the odd-degree vertices is 4,which is even as per the corollary of handshaking theorem.

Now, the sum of degrees of all vertices is 2e, where is the number of edges. So, we get

3+3+3+2+2+1=2e

2e =14

e= 7

Hence, the number of edges in the given graph is 7and can be verified by counting.

Example 5.7 :

For each of the following degree sequences, determine if there exists a graph whose degree sequence is given. If possible draw the graph or explain why such a graph does not exist.

i. (1, 1, 1, 1, 1)

ii. (1, 1, 1, 1, 1, 1)

Solution

(i) The given degree sequence is (1, 1, 1, 1, 1).

Sum of the degrees of the vertices =1 + 1 + 1 + 1 + 1 = 5 = odd number.

Hence, it is not possible to draw any *graph corresponding to the degree* sequence (*1*,*1*,*1*, *1*,*1*).

(ii) The given degree sequence is (1, 1, 1, 1, 1, 1).

Sum of degrees = 1 + 1 + 1 + 1 + 1 + 1= 6 = even number

Therefore, e = number of edges = 6/2 = 3.

Here, n (number of vertices in the graph) = *6*.

Also, number of odd-degree vertices *is 6 and it is an even number.*

Hence, the graph corresponding to the given sequence (1,1,1,1,1,1) can be drawn.

Example 5.8 :

Let G be a simple graph with 12 edges. If G has 6 vertices of degree 3 and the rest of the vertices have degree less than 3, then find the (a) minimum number of vertices and (b) maximum number of vertices.

Solution:

Number of edges e = 12

Suppose the total number of vertices in *G* is *p*.

Given that 6 vertices have degree 3.

Hence, the sum of degrees =3 *6 = 18.

The rest *(p* — 6) vertices have degree less than 3. i.e., their degree lies inclusively between 0 and 2.

Here, to find the *minimum* number of vertices, *(p* — 6) vertices must have *maximum* degree [i.e., 2]

Therefore, applying the handshaking theorem, we get

$$18 + 2(p - 6) = 2e$$

$$18 + 2p - 12 = 24$$

$$2p + 6 = 24$$

$$2p = 18$$

$$\Rightarrow p = 9$$

Minimum number of vertices = 9

To calculate the *maximum* number of vertices, *(p* — 6) vertices must have *maximum* degree [i.e., 1]

Sum of degrees = $18 + (p - 6) = 2e = 24$

$$p + 12 = 24$$

$$\Rightarrow p = 12$$

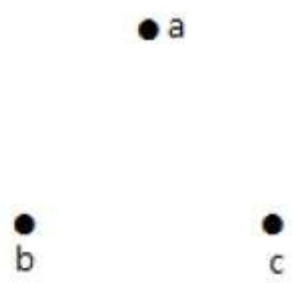
Maximum number of vertices = 12

5.2 TYPES OF GRAPHS

Some important types of graphs are introduced here. These are often used in many applications.

5.2.1 Null Graph

A graph which contains only isolated nodes is called a null graph. i.e the set of edges in a null graph is empty. Null graph on n vertices in denoted by N_nNull graph (N_3) with 3 vertices is shown below



5.2.2 Complete Graph

A graph G is said to be complete if every vertex of G is connected with every other vertex of G. i.e., every pair of distinct vertices contains exactly one edge. Complete graph on n vertices is denoted by K_n.

Some complete graphK_1, K_2, K_3, K_4, K_5, K_6, $K_7$are shown below.

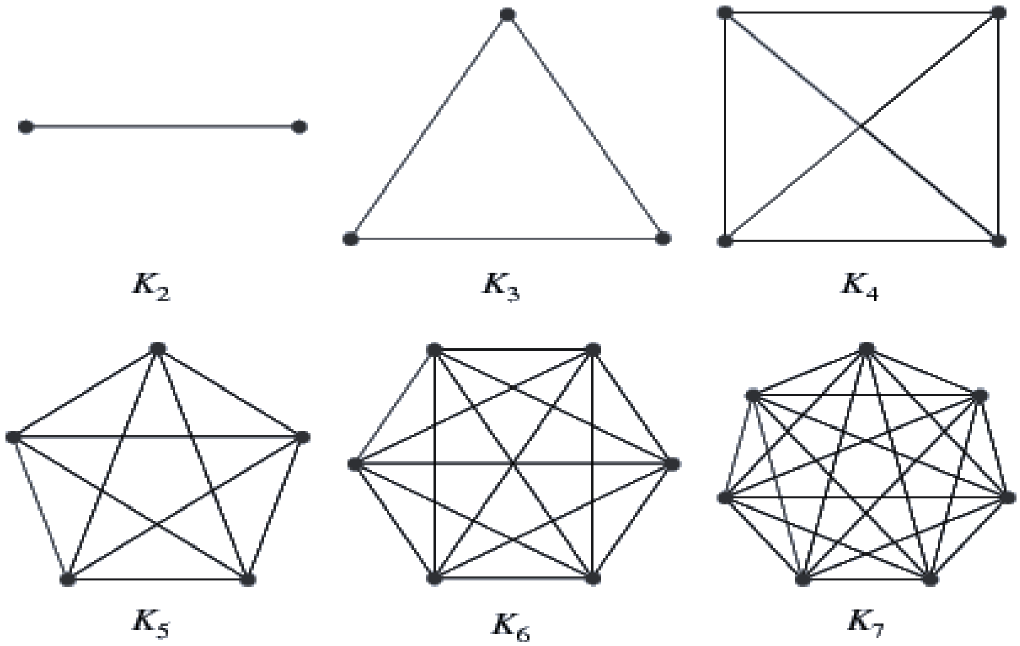


Figure 5.9

A complete graph G is a simple graph and it may be *directed* as well *as* undirected. Any complete graph K_n with n vertices has exactly $n(n — 1)/2$ edges.

Directed graph K_3 is shown in Figure 5.10.

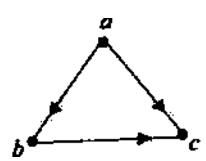


Figure 5.10

5.2.3 Regular Graph

A graph in which all the vertices are of *same degree is called a regular graph. If the degree* each vertex is *r,* then the graph is called a regular graph of degree r, and it is denoted byR_r. A regular graph may be *directed* or *undirected.* When it is directed, *then the degree of each vertex is computed* as *the* sum of its *in-degree* and *out- degree.*

A complete graph is a regular graph of degree n-1 or it is called (n-1) regular graph. Obviously, if a graph is null graph, then it is 0 regular (as degree of each vertex is 0)

2-regular graph with 5 vertices is given in Figure 5.11

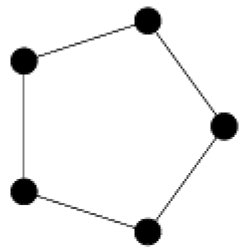


Figure 5.11

Note:

If a graph G with n vertices is r-regular, then the number of edges of G is

r * n/2.

Since the graph has n vertices n vertices are r-regular, the sum of the degree of the vertices is n * r. Also, sum of degrees of a graph equals to twice the number of edges. Hence, the number of the edges of the regular graph with n vertices is r * n/2.

Example 5.9

Find the number of edges of a 4-regular graph with 6 vertices.

Solution :

Here n = 6 and r = 4.

Number of edges $(e) = r * \frac{n}{2} = 4 * \frac{6}{2} = 12$

Example 5.10

Is it possible to draw a 3-regular graph with 5 vertices.

Solution :

Number of vertices n = 5

r= 3

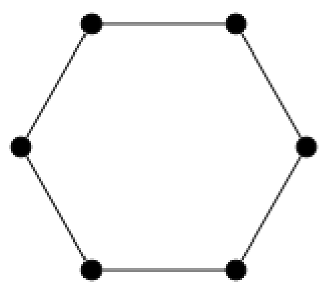
Sum of the degrees of the vertices = 5 * 3 = 15, *which is not divisible* by 2. Therefore, it is not possible to draw a 3-regular with 5 vertices.

Note :A graph with n vertices is r-regular if either r or n or both are even.

Cycles

The cycle C_n , $n \geq 3$, consists of n vertices and n edges so that the second endpoint of the last edge coincides with the staring vertex.

A cycle with 6 vertices is shown below.

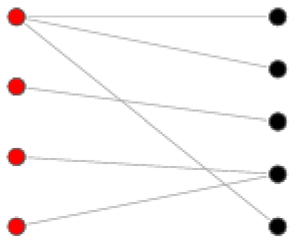


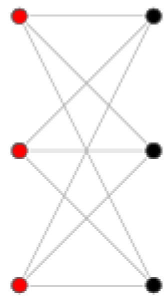
5.2.4 Bipartite Graph

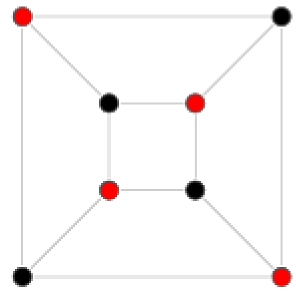
A graph *G* = (V, *E*) is a bipartite graph if the vertex set *V* can be partitioned into two disjoint subsets, say, V_1 and V_2such that every edge in *E* connects a vertex in $V_1$to the vertex in V_2.

But no edge in G connects either of the two vertices in V_1 or two vertices in V_2.

(V_1, V_2) is called a *bipartition* of G. Some examples of bipartite graph are shown below.



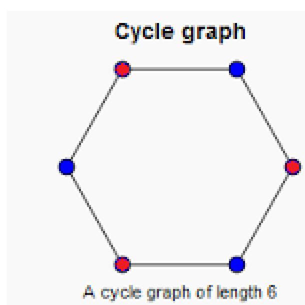




Example 5.11

Show that the graph C_6 is bipartite.

Solution:



A cycle graph of length 6

In this graph, the two distinct sets of vertices are shown in distinct colours. Hence, C_6 is bipartite.

Procedure to check whether a graph G is bipartite or not

Step 1 Arbitrarily select a vertex from G and include it into set 1.

Step 2 Consider the edges directly connected to that vertex and put the other end vertices of these edges into set 2.

Step 3 Now, pick up one vertex from set 1and consider the *edges* directly connected to that vertex, and put the other end vertices of these edges into set 2.

Step 4 At each step, step 2 and step 3, check if there is any edge among the vertices of set 1 or set 2.

If so, construction of sets is stopped and the given graph is not bipartite graph, then return.

Else continue step 2 and step 3 alternately until all the vertices are included in union of set 1 and set 2.

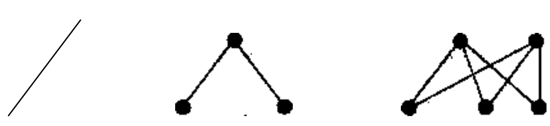
Step 5 If two computed sets following the above steps are distinct, then it is bipartite.

5.2.5 Complete Bipartite Graph:

A bipartite graph *G* is a complete bipartite graph if there is an edge between every pair of vertices taken from two disjoint sets of vertices (one vertex from one set$V_1$and the other from set V_2).

Complete bipartite graph G is denoted by K_{mn}, where *m* and *n* are the number of vertices in two distinct subsetsV_1 and V_2.

Some examples of complete bipartite graphs are shown in Figure.



(a) $K_{1,1}$ (b) $K_{1,2}$ (c) $K_{2,3}$ (d) $K_{2,2}$

Example 5.12

How many edges do the complete bipartite graph, $K_{m\ n}$ have?

Solution:

The vertex set of $K_{m\ n}$ consists of two disjoint sets A and *B*.

A contains *m* vertices and *B* contains *n* vertices.

Each vertex in A is adjacent to each vertex in *B.*

No two vertices either in A or in *B* are adjacent.

Hence, the degree of each vertex in *A* is *n*, and the degree of each vertex in *B* is m.

Therefore, the sum of the degrees is 2 * *m** *n*, and so there are *m** *n* edges (as per the *handshaking* theorem).

Note : Complete bipartite graph $K_{m,n}$ has m + n vertices and m * n edges.

Km, n is regular if m= n.

Example 5.13

Prove that a graph which contains a triangle cannot be bipartite.

Solution

In a bipartite graph, the vertices should be divided into two distinct subsets.

The number of vertices of the given graph is 3, as it is a triangle. So, it is not possible to divide the vertices into two disjoint set of vertices since each edge is joined by the rest two edges.

Hence, this graph may not be a bipartite graph.

5.3 SUBGRAPH

If G and H are two graphs with vertex sets V(G*)* and V(H) and edge sets E(G*)* and E(H*)*,respectively, such that V(H)⊆ V(G) and E(H)⊆E(G*)*, then we say that *H* is a subgraph of *G*or G is a super-graph of *H.*

In other words, if *H* is a subgraph of G, then all the vertices and the edges of *H* are in G and each edge of *H* has the same endpoints as in *G.*

Now if V(H)= V(G) *and* E(H) C E(G), then we say that *H* is a ***spanning subgraph*** of G.

A *spanning subgraph* is a subgraph that contains all the vertices of the original graph.

If H is a subgraph of G, then

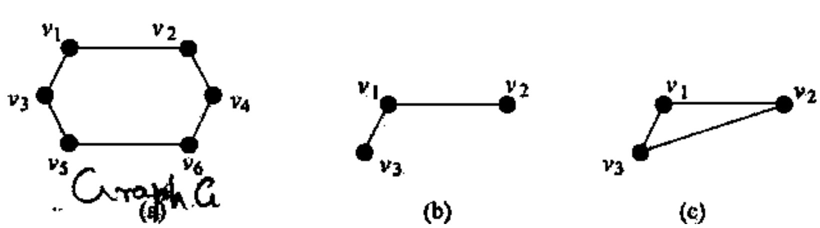
(a) All the vertices of H are in G.

(b) All the edges of H are in G.

(c) Each edge of H has the same endpoints in *H* as in *G.*

For example

A graph G is shown below and its one subgraphis in Figure (b), but the graph shown in Figure (c) is not a subgraph of *G,* as no edge between v_3 and v_2 is present in the original graph G.



Note :

Suppose a graph G has n number of vertices (i.e., |V| = n) and m number of edges (i.e., |E|=n)

Then, number of non-empty *subsets* of V as 2^n - 1 and number of subsets of E as 2^m.

Thus, the total number of non-empty subgraphs of G is $(2^n - 1) * 2^m$.

Example 5.14

Prove that the number of spanning subgraphs of a graph G with m vertices is 2^m

Proof :

Number of spanning subgraphs with 0 (zero) edge and m vertices is ^mC_o

Number of spanning subgraphs with 1 edge and m vertices is $^mC_1$.

Number of spanning subgraphs with 2 edges and m vertices is $^mC_2$.

….. ….

Number of spanning subgraphs with r edges and m vertices is ^mC_r.

……. ……..

Number of spanning subgraphs with m edges and m vertices is ^mC_m.

Total number of spanning subgraphs

$$= {}^mC_o + {}^mC_1 + {}^mC_2 + \bullet \bullet \bullet + {}^mC_m$$

$$= 2^m \text{ (by binomial theorem)}$$

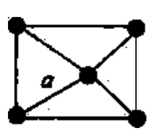
Example 5.15

For the graph G draw the subgraphs

(a) $G — e$ (here, e is one edge)

(b) $G — a$ (here, a is one vertex)

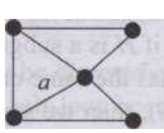
e

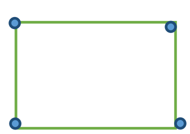


G

Solution :

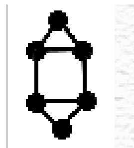
The subgraphs are shown below



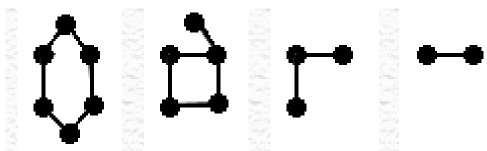


G –e **G – a**

Example 5.16 : Draw some subgraphs of the graph



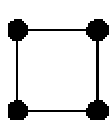
Solution :



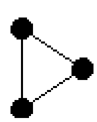
5.4 OPERATIONS ON GRAPH

In this section some operation on graph are discussed.

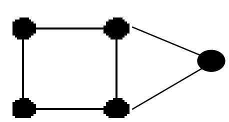
i. **Union of two graphs** G1 and G2 will be another graph G such that V(G1∪G2) = V(G1) ∪V(G2) and E(G1∩G2)= E(G1) ∩E(G2)
If no common vertex is present in between G1 and G2 then the resultant graph will be disconnected.



G1



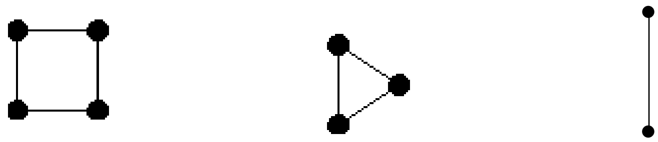
G2



G1 U G2

ii. **Intersection of two graph** G1 and G2 will be another graph G such that

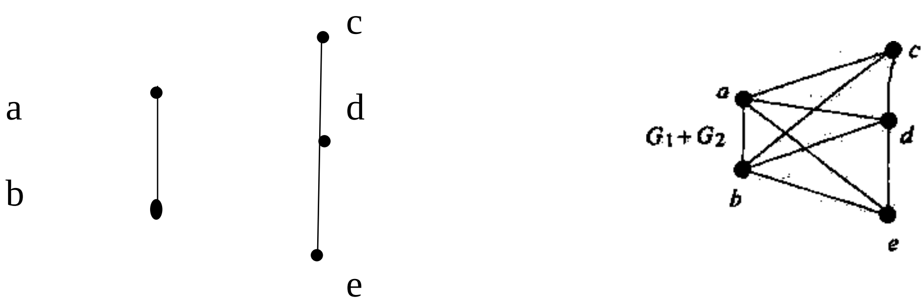
$V(G_1 \cap G_2) = V(G_1) \cap V(G_2) \neq \Phi and E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$



G1 G2 G1 ∩G2

iii. **Sum of two graphs**

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_1)$ be two graphs such that $V_1 \cap V_2 \neq \Phi$. The sum of two graphs G_1 and G_2 is G1+G2 is defined as the graph G in which vertex set is V_1+V_2 and the edge set consists of the edges in E_1 and E_2 and the edges joining each vertex of V_1 with each vertex of V_2.



G_1 G_2 $G_1 + G_2$

iv. ***Complement***: The complement *G'* of *G* is defined as a simple graph (parallel edge and self-loop are ignored) with the same vertex set as *G, and where two vertices u* and v are adja*cent only when they are not adjacent in G.*

G G'

v. **Product of two graphs** G_1 and G_2 is defined as G = (V_1 *U* V_2, V_1x

V_2)where $V_1 \cup V_2$ is the union of the vertex sets V_1 of G_1 and $V_2$of G_2, and V_1 x V_2 is the *cross product* to compute the edge set of the resultant graph G.

5.5 REPRESNTATION OF GRAPH

Diagrammatic (graphical) representation of a graph is very *convenient for* visual study, but it is practically feasible only when the number of vertices and edges of the graph is reasonably small. So, we need some other reasonable ways to represent graphs with large number of vertices and edges. These *representations are* also expected to be useful in computer programming. Some representations for undirected as well as directed graphs are discussed below.

5.5.1 Matrix (Adjacency Matrix) Representation

The adjacency matrix is commonly used to represent graphs for computer processing. In such representation, an n x n *Boolean* (1,0) matrix is used where a 1 at position (u, v) indicates that there exists an edge from vertex *u* to v, and a 0 at position(u, v) indicates that there is no edge reachable directly from *u* to v.

If the graph is undirected, then its corresponding adjacency matrix will be symmetric.

(i) Matrix presentation of undirected graph

If an undirected graph G consists of n vertices (assuming that the graph has no parallel edge), then the adjacency matrix of G is an n x n matrix A =$\left[a_{ij}\right]$ and defined as follows:

$$a_{i,j} = \begin{cases} 1, if\ there\ is\ an\ undirected\ edge\ between\ vi\ and\ vj \\ 0, \quad if\ there\ is\ no\ edge\ between\ vertices\ vi\ and\ vj \end{cases}$$

Some observations from matrix representation of undirected simple graph:

(a) $a_{i,j} = a_{j,i}$ for all i and j, i.e., the matrix is symmetric.

(b) Diagonal elements of the matrix are zero (0) (as the simple graph possesses no self loop).

(c) The degree of a vertex is the sum of the 1s in that row.

(d) Let G be a graph with n vertices: V_1, V_2, V_3,…,V_n and A be the adjacency matrix of G. Let B be the matrix computed as follows:

$$B = A + A^2 + A^3 + \cdots + A^n (n > 1)$$

Now, B is connected if and only if B has no zero entry.

(ii) **Matrix representation of directed graph**

Let G a be a directed graph (digraph) consists of n vertices (assuming that the graph has no parallel edge. The adjacency matrix of G is an n x matrix A $=[a_{i,j}]$ and is defined as follows

$a_{i,j} = \{1,$ if there is a directed edge between vertices v_i and v_j

0, otherwise

Some observations from the matrix representation of directed simple graph.

a. $a_{i,j} \neq a_{j,i}$ for all i and j
b. Diagonal elements of the matrix A are 0
c. The sum of 1 in any column j of A is equal to the in-degree of vertex v_j.
d. The sum of 1 in any row i of A is equal to the out-degree of vertex v_i

5.5.2 Incidence Matrix Representation of Graph

Let G be a graph with n verities and e edges.

The incidence matrix is defined as an nxe matrix $B=[b_{i,j}]$ where

$$b_{i,j} = \begin{cases} 1, & \textit{if jth edge ej is incident with the ith vertx} \\ 0, & \textit{otherwise} \end{cases}$$

Example 5.17 :

Write the incidence matrix of the graph G given in figure

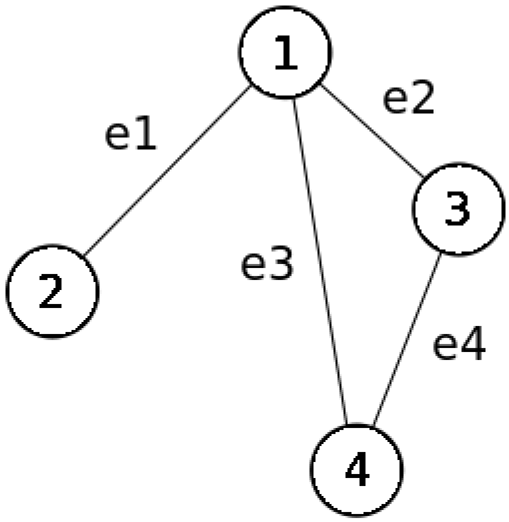


Figure 5. 13

Solution :

The incidence matrix is

Edge / Vertex	e1	e2	e3	e4
Vertex 1	1	1	1	0
2	1	0	0	0
3	0	1	1	1
4	0	0	1	1