Evaluation of Integrals using Residue Theorem

We use the following theorems to find the residues of $f(z)$

Theorem 1:

Let $f(z) = \frac{\phi(z)}{(z-a)^m}$, m is a +ve integer and $\phi(z)$ is analytic in the neighbourhood of a which is a pole of order m

Then $\underset{z=a}{\text{Res}}\, f(z) = \frac{\phi^{(m-1)}(a)}{(m-1)!}$

Theorem 2: Consider the case of a simple pole (pole of order 1)

Let $f(z) = \frac{\phi(z)}{z-a}$, where $\phi(z)$ is analytic in the neighbourhood of a.

Then $\underset{z=a}{\text{Res}}\, f(z) = \lim_{z \to a} (z-a) f(z)$

Example: Find the poles and Residues of $\frac{1}{z^2+5z+6}$

Let $f(z) = \frac{1}{z^2+5z+6}$

$= \frac{1}{(z+3)(z+2)}$

$\therefore$ $f(z)$ has -3 and -2 as poles of order 1

when $a = -3$

$f(z) = \frac{\phi(z)}{z+3}$ where $\phi(z) = \frac{1}{z+2}$

$$\underset{z=-3}{Res} f(z) = \underset{z\to -3}{lt} (z+3) f(z)$$

$$= \underset{z\to -3}{lt} (z+3) \frac{1}{(z+3)(z+2)} = -1$$

when $a = -2$

$$\underset{z\to -2}{Res} f(z) = \underset{z\to -2}{lt} (z+2) f(z) = 1$$

Evaluation of Definite Integrals

Type I: The integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$. where the integral is a rational function of $\cos\theta$ and $\sin\theta$.

put $z = e^{i\theta}$; $dz = ie^{i\theta} d\theta$

$dz = iz d\theta$

$d\theta = \frac{dz}{iz}$.

$\cos\theta = \frac{1}{2}(z + 1/z)$

$\sin\theta = \frac{1}{2i}(z - 1/z)$

Then the integral becomes

$$\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta = \frac{1}{i} \int_c f\left(\frac{1}{2}(z+1/z), \frac{1}{2}(z-1/z)\right) \frac{dz}{z}$$ where c is the unit circle $|z|=1$

$$= \int_c \phi(z) dz$$

clearly $\phi(z)$ is a rational function of z

Then by residue theorem,

$\int \phi(z) dz = 2\pi i \times$ sum of residues of $\phi(z)$ at its poles inside c.

1) Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ where $a>b>0$

Let $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$

put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}(z+1/z)$

$\therefore I = \frac{1}{i}\int_C \frac{dz}{z(a+b/2(z+\frac{1}{z}))}$

$= \frac{1}{i}\int_C \frac{dz}{z\left(\frac{2za+b(z^2+1)}{2z}\right)}$

$= \frac{2}{i}\int_C \frac{dz}{2za+bz^2+b} = \int_C \phi(z)\,dz$

where $\phi(z) = \frac{2}{i(z^2b+2za+b)}$

To find the poles of $\phi(z)$

It is enough to find the zeros of $bz^2+2az+b=0$.

$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$

$= \frac{-a \pm \sqrt{a^2-b^2}}{b}$

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

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Since $a > b > 0$, $|\beta| > 1$ and $|\alpha| < 1$

Hence $z = \alpha$ is the only simple pole lies inside C.

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z \to \alpha} (z-\alpha)\phi(z)$$

$$= \lim_{z \to \alpha} (z-\alpha) \frac{2}{i(z-\alpha)(z-\beta)b}.$$

$$= \lim_{z \to \alpha} \frac{2}{i(z-\beta)b}$$

$$= \frac{2}{i(\alpha-\beta)b} = \frac{2}{ib\left(\frac{2\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{1}{i\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = 2\pi i \times \text{sum of residues of } \phi(z) \text{ at its poles inside } C.$$

$$= 2\pi i \times \frac{1}{i\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Type 2: Integrals of type $\int_{-\infty}^{\infty} f(x)\,dx$.

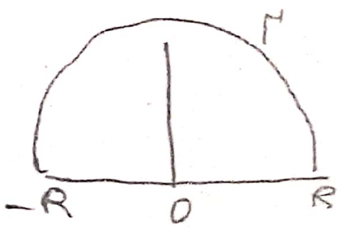
Theorem: If $f(z)$ is a function which is analytic in the upperhalf plane except at a finite no. of poles in it having no poles on the real axis and if further $z f(z)$ tends to zero as $|z| \to \infty$ then by contour integration $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \times$ Sum of residues at its poles in the upper half plane.

Lemma: If c is an arc, $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$ and if $\lim_{z\to\infty} z f(z) = A$ then $\lim_{R\to\infty} \int_c f(z)\,dz = (\theta_2 - \theta_1)A$

Problem: Prove that $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \pi/4$

consider $\int_c \frac{dz}{(1+z^2)^2} = \int_c f(z)\,dz$

where c is the contour consisting of the large semicircle Γ of radius R together with the part of the real axis from $-R$ to R



Then by Residue theorem,

$\int_c f(z)\,dz = 2\pi i \times$ sum of the residues at poles inside c

i.e. $\int_c \frac{dz}{(1+z^2)^2} = 2\pi i \times$ Sum of the residues

$$\Rightarrow \int_{-R}^{R} \frac{dx}{(1+x^2)^2} + \int_{\Gamma} \frac{dz}{(1+z^2)^2} = 2\pi i \times \text{sum of the residues at its poles inside } C \quad \text{(1)}$$

Now, $\lim_{z\to\infty} z \cdot f(z) = 0$.

$\therefore$ By the lemma, $\lim_{R\to\infty} \int_{\Gamma} \frac{dz}{(1+z^2)^2} = 0$.

From (1), $\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(1+x^2)^2} = 2\pi i \times$ sum of residues at its poles inside upper half plane

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \times \text{sum of residues at its poles in the upper half plane} \quad \text{(2)}$$

<u>To find the poles and residues</u>.

$f(z) = \frac{1}{(1+z^2)^2}$ has a pole of order 2 at $z = \pm i$

out of these 2 poles $z = \pm i$, $z = i$ is the only pole lie in the upper half plane.

$\therefore \operatorname{Res}_{z=i} f(z) = \frac{\phi'(i)}{1!} = \phi'(i)$, where $\phi(z) = \frac{1}{(z+i)^2}$

$\phi'(z) = \frac{-2}{(z+i)^3}$

$$= \frac{1}{4i}$$

From (2) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \times \frac{1}{4i} = \pi/2$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{(1+x^2)^2} = \pi/2$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{(1+x^2)^2} = \pi/4$$

Type 3: Evaluation of integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx \, dx, \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx \, dx \text{ where}$$

i) $P(x)$ and $Q(x)$ are polynomials.

ii) degree of $Q(x)$ exceeds that of $P(x)$

iii) The equation $Q(x) = 0$ has no real roots.

To evaluate this type, we need the following Jordan's lemma.

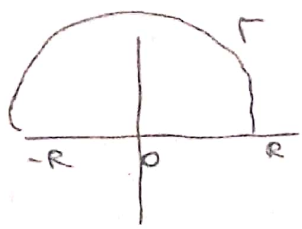
Jordan's lemma: If $f(z) \to 0$ uniformly as $z \to \infty$ and $f(z)$ is meromorphic in the upper half plane then

$$\lim_{R \to \infty} \int_{\Gamma} e^{imz} f(z) \, dz = 0 \quad (m > 0) \text{ where } \Gamma \text{ denotes the}$$

semi circle $|z| = R$ and $Im(z) > 0$.

Problem 1. Prove that $\displaystyle\int_{0}^{\infty} \frac{\cos x}{x^2 + a^2} = \frac{\pi}{2a} e^{-a}$

Solution: Consider $\displaystyle\int_C \frac{e^{iz} dz}{z^2 + a^2} = \int_C e^{iz} f(z) \, dz$ where



C is a contour as shown in the figure

By Residue theorem,

$$\int_{-R}^{R} e^{ix} f(x) \, dx + \int_{\Gamma} e^{iz} f(z) \, dz = \int_C e^{iz} f(z) \, dz$$

$= 2\pi i \times$ sum of residues at poles in the upper half plane $\rightarrow (1)$

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Now, $\lim_{z\to\infty} \frac{1}{z^2+a^2} = 0$

$\therefore$ By Jordan's Lemma, $\lim_{R\to\infty} \int_{\Gamma} \frac{e^{iz}\,dz}{z^2+a^2} = 0$

i.e) $\lim_{R\to\infty} \int_{\Gamma} e^{iz} f(z)\,dz = 0$

$\therefore$ (1) becomes $\int_{-\infty}^{\infty} e^{ix} f(x)\,dx = \int_C e^{iz} f(z)\,dz = 2\pi i \times$ sum of residues at poles in the upper half plane

i.e) $\int_{-\infty}^{\infty} e^{ix} f(x)\,dx = 2\pi i \times$ sum of residues at poles in the upper half plane $\rightarrow$ (2)

To find the poles and residues

The poles of $f(z)$ is given by $z^2+a^2=0 \Rightarrow z = \pm ia$.

Out of these two poles $z=ai$ is the pole lie in the upper half plane

$\therefore \operatorname{Res}_{z=ai} f(z) = \lim_{z\to ai} (z-ai)\, e^{iz} f(z) = \lim_{z\to ai} \frac{(z-ai)\, e^{iz}}{(z+ai)(z-ai)} = \frac{e^{-a}}{2ai}$

From (2) $\int_{-\infty}^{\infty} e^{ix} f(x)\,dx = 2\pi i \times \frac{e^{-a}}{2ai} = \frac{e^{-a}\pi}{a}$

i.e) $\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}$

Equating real part we get,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} = \frac{\pi e^{-a}}{a}$$

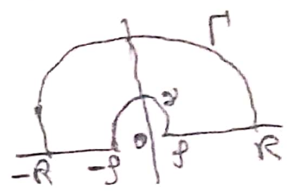
$$\int_{0}^{\infty} \frac{\cos x}{x^2+a^2} = \frac{\pi e^{-a}}{2a}.$$

Type 4: Integrals of many valued functions:

In the case of integrals involving many valued functions such as $\log z$, z^a where a is not an integer, we should use only those contours whose interiors do not contain any branch points and particular branches should also be specified.

Problem: Evaluate $\int_0^\infty \frac{x^{1/3}}{1+x^2} dx$.

Conside $\int_C \frac{z^{1/3}}{1+z^2} dz = \int_C f(z)dz$ where C is the contour consisting of a large semicircle in the upper half plane intented at the origin.



Here we have avoided the branch point 0 of $z^{1/3}$ by intending the origin.

The poles of $f(z)$ is given by $1+z^2=0 \Rightarrow z = \pm i$

Out of these two poles $z=i$ is the only simple pole lies within C

$$\therefore \operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} (z-i) \frac{z^{1/3}}{(z+i)(z-i)} = \frac{i^{1/3}}{2i}$$

$$= \frac{(\cos \pi/2 + i \sin \pi/2)^{1/3}}{2i} = \frac{\cos \pi/6 + i \sin \pi/6}{2i}$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2i}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right).$$

By Residue theorem,

$$\int_C f(z)dz = \int_\rho^R f(x)dx + \int_\Gamma f(z)dz + \int_{-R}^{-\rho} f(x)dx + \int_\gamma f(z)dz$$

$$= 2\pi i \times \text{sum of the residues}$$

$$\Rightarrow \int_{\rho}^{R} f(x)dx + \int_{\Gamma} f(z)dz + \int_{R}^{\rho} f(xe^{i\pi})e^{i\pi}dx + \int_{\gamma} f(z)dz = 2\pi i \times \frac{1}{2i}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$$
$$= \pi\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) \quad \to (1)$$

Now

$$\left|\int_{\Gamma} f(z)dz\right| \le \int_{\Gamma} \left|\frac{z^{1/3}}{1+z^2}\right| dz$$
$$\le \int_0^{\pi} \frac{|Re^{i\theta}|^{1/3}}{|1+R^2e^{2i\theta}|} \times |Re^{i\theta} i|\,d\theta, \quad z = Re^{i\theta}, \ dz = iRe^{i\theta}d\theta, \ 1+z^2 = 1+R^2e^{2i\theta}$$
$$\le \int_0^{\pi} \frac{R^{1/3}}{1+R^2} \cdot R\,d\theta.$$
$$= \int_0^{\pi} \frac{R^{4/3}}{1+R^2} d\theta \to 0 \text{ as } R \to \infty.$$

Also

$$\left|\int_{\gamma} f(z)dz\right| \le \int_{\gamma} |f(z)||dz|$$
$$\le \int_{\gamma} \frac{|z^{1/3}|}{(1+z^2)} |dz| \qquad z = \rho e^{i\theta}, \ dz = i\rho e^{i\theta}d\theta$$
$$= \int_{\gamma} \frac{\rho^{4/3}}{\rho^2+1}\pi \to 0 \text{ as } \rho \to 0$$

$$\therefore \lim_{\rho\to 0} \int_{\gamma} f(z)dz = 0$$

Hence as $R \to \infty$ and $\rho \to 0$ from (1) we get

$$\int_0^{\infty} f(x)dx + \int_{\infty}^{0} f(xe^{i\pi})e^{i\pi}dx = \pi\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$$

$$\therefore \int_0^{\infty} \frac{x^{1/3}}{1+x^2}dx + \int_{\infty}^{0} f(xe^{i\pi})(-dx) = \pi\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$$

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i.e) $\int_0^\infty \frac{x^{1/3}}{1+x^2}dx + \int_0^\infty f(xe^{i\pi})dx = \pi\left(\sqrt{3}/2 + i/2\right)$

i.e) $\int_0^\infty \frac{x^{1/3}}{1+x^2}dx + \int_0^\infty \frac{x^{1/3}e^{i\pi/3}}{1+x^2e^{2i\pi}}dx = \pi\left(\sqrt{3}/2 + i/2\right)$

i.e) $\int_0^\infty \frac{x^{1/3}}{1+x^2}dx + \int_0^\infty \frac{x^{1/3}(\cos\pi/3 + i\sin\pi/3)}{1+x^2}dx = \pi\left(\sqrt{3}/2 + i/2\right)$

i.e) $\int_0^\infty \frac{x^{1/3}(\cos\pi/3 + 1 + i\sin\pi/3)}{1+x^2}dx = \pi\left(\sqrt{3}/2 + i/2\right)$

Equating the real part we get

$$\int_0^\infty \frac{x^{1/3}(\cos\pi/3 + 1)}{1+x^2}dx = \pi\left(\sqrt{3}/2\right)$$

i.e) $(1+\cos\pi/3)\int_0^\infty \frac{x^{1/3}}{1+x^2}dx = \pi\sqrt{3}/2.$

i.e) $\int_0^\infty \frac{x^{1/3}}{1+x^2}dx = \frac{\pi\sqrt{3}}{2} \times 2/3 = \frac{\pi}{\sqrt{3}}$

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}dx = \pi/\sqrt{3}.$$

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